Attainable Sets of the Nonlinear Control Systems with Integral Constraint on Controls

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Consider the control system the behavior of which is described by the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) \in X_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the phase state vector of the system, $u \in \mathbb{R}^m$ is the control vector, $t \in [t_0, \theta]$ is the time and $X_0 \subset \mathbb{R}^n$ is a compact set.

By $L_p\big([t_0, \theta], \mathbb{R}^m\big)$ $(p > 1)$ we denote the space of measurable functions $u(\cdot) : [t_0, \theta] \to \mathbb{R}^m$ with finite $\|u(\cdot)\|_p$ norm where $\|u(\cdot)\|_p = \left(\int\limits_{t_0}^{\theta} \|u(t)\|^p \, dt \right)^{\frac{1}{p}}$ and $\|\cdot\|$ denotes the Euclidean norm.

For $p \in (1, \infty)$ and $\mu_0 > 0$ we set

$$U_p = \left\{ u(\cdot) \in L_p\big([t_0, \theta], \mathbb{R}^m\big) : \|u(\cdot)\|_p \leq \mu_0 \right\} \tag{2}$$

where μ_0 defines the resource of the control effort.

Every function $u(\cdot) \in U_p$ is said to be an admissible control function. It is obvious that the set of all admissible control functions U_p is the closed ball centered at the origin with radius μ_0 in $L_p\big([t_0, \theta]; \mathbb{R}^m\big)$.

It is assumed that the right hand side of the system (1) satisfies the following conditions:

1.A. The function $f(\cdot) : [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous;

1.B. For any bounded set $D \subset [t_0, \theta] \times \mathbb{R}^n$ there exist constants $L_1 = L_1(D) > 0$, $L_2 = L_2(D) > 0$ and $L_3 = L_3(D) > 0$ such that

$$\|f(t, x_1, u_1) - f(t, x_2, u_2)\| \leq [L_1 + L_2 (\|u_1\| + \|u_2\|)] \|x_1 - x_2\| + L_3 \|u_1 - u_2\|$$

for any $(t, x_1) \in D$, $(t, x_2) \in D$, $u_1 \in \mathbb{R}^m$ and $u_2 \in \mathbb{R}^m$;

1.C. There exists a constant $c > 0$ such that

$$\|f(t, x, u)\| \leq c(1 + \|x\|)(1 + \|u\|)$$

for every $(t, x, u) \in [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m$.

Note that if the norm of the control effort is large then the conditions 1.B and 1.C permit the system to assume high velocities.

Let the right hand side of the system (1) be affine, i.e. $f(t, x, u) = \varphi(t, x) + B(t, x)u$ and the functions $\varphi(\cdot) : [t_0, \theta] \times \mathbb{R}^n \to \mathbb{R}^n$, $B(\cdot) : [t_0, \theta] \times \mathbb{R}^n \to \mathbb{R}^m$ satisfy the following assumptions:

1.D. The functions $f(t, x)$ and $B(t, x)$ are continuous on (t, x) and for any bounded set $D \subset [t_0, \theta] \times R^n$ there exist Lipschitz constants $\kappa_i = \kappa_i(D) \in (0, \infty)$ $(i = 1, 2)$ such that

$$\|\varphi(t, x_2) - \varphi(t, x_1)\| \leq \kappa_1 \|x_2 - x_1\|, \quad \|B(t, x_2) - B(t, x_1)\| \leq \kappa_2 \|x_2 - x_1\|$$

for any $(t, x_1) \in D$, $(t, x_2) \in D$.

1.E. There exist constants $\nu_i \in (0, \infty)$ $(i = 1, 2)$ such that

$$\|\varphi(t, x)\| \leq \nu_1 (1 + \|x\|), \ \|B(t, x)\| \leq \nu_2 (1 + \|x\|)$$

for every $(t, x) \in [t_0, \theta] \times R^n$.

Then it is possible to verify that the function $f(t,x,u)=\varphi(t,x)+B(t,x)u$ satisfy the conditions 1.A, 1.B and 1.C.

Let $u_*(\cdot)\in U_p$. The absolutely continuous function $x_*(\cdot):[t_0,\theta]\to\mathbb{R}^n$ which satisfies the equation $\dot{x}_*(t)=f(t,x_*(t),u_*(t))$ a.e. in $[t_0,\theta]$ and the initial condition $x_*(t_0)=x_0\in X_0$ is said to be a solution of the system (1) with initial condition $x_*(t_0)=x_0$, generated by the admissible control function $u_*(\cdot)$. By the symbol $x(\cdot;t_0,x_0,u(\cdot))$ we denote the solution of the system (1) with initial condition $x(t_0)=x_0$, which is generated by the admissible control function $u(\cdot)$. Note that the conditions 1.A - 1.C guarantee the existence, uniqueness and extendability of the solutions up to the instant of time θ for every given $u(\cdot)\in U_p$ and $x_0\in X_0$.

Let us define the sets

$$Y_p(t_0,X_0)=\left\{x\left(\cdot;t_0,x_0,u(\cdot)\right):x_0\in X_0,\ u(\cdot)\in U_p\right\},$$

$$X_p(t;t_0,X_0)=\{x(t)\in\mathbb{R}^n:x(\cdot)\in Y_p(t_0,X_0)\}$$

where $t\in[t_0,\theta]$.

The set $X_p(t;t_0,X_0)$ is called the attainable set of the system (1) at the instant of time t. It is obvious that the set $X_p(t;t_0,X_0)$ consist of all $x\in\mathbb{R}^n$ to which the system (1) can be steered at the instant of time $t\in[t_0,\theta]$.

In this presentation various topological properties of the attainable sets $X_p(t;t_0,X_0)$, $t\in[t_0,\theta]$, of the control system (1) are investigated. An approximation method has been obtained for numerical construction of the attainable sets.