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LYAPUNOV EXPONENTS FOR CONTINUOUS TRANSFORMATIONS AND DIMENSION THEORY

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Abstract. We generalize the concept of Lyapunov exponent to transformations that are not necessarily differentiable. For fairly large classes of repellers and of hyperbolic sets of differentiable maps, the new exponents are shown to coincide with the classical ones. We also discuss the relation of the new Lyapunov exponents with the dimension theory of dynamical systems for invariant sets of continuous transformations.

1. Introduction

The notion of Lyapunov exponent has its origins in the pioneering work of Lyapunov and can be introduced in the following manner. Given a differentiable transformation $f\colon M \to M$, for each $x \in M$ and $v \in T_xM$ we define the *Lyapunov exponent* of the pair (x, v) by

$$\lambda(x,v) = \limsup_{n\to+\infty} \frac{1}{n} \log \|df_x^n v\|.$$

These numbers provide precious information about the asymptotic behavior of the differential of f along the orbit of x, and namely about the expansion and contraction of lengths on the tangent space. Hence, there is a privileged relation between the Lyapunov exponents and the study of hyperbolicity. Furthermore, there is also a profound relation, although hardly obvious, between them and the stability theory of differential equations and dynamical systems. In particular, the Lyapunov exponents—including the "abstract" theory of regularity, which is not so well known, even though its origins go back to Lyapunov and Perron—play a preeminent role in the stability theory of nonautonomous ordinary differential equations and in the invariant manifold theory of nonuniformly hyperbolic dynamical systems (see Section 2.2; see also [2] for full details). On the other hand, without any change, the fact that the derivative appears in the definition of the Lyapunov exponents restricts their use to the study of differentiable transformations.

Our main objective is to discuss how the concept of Lyapunov exponent can be effectively generalized—and applied—to a fairly general class of transformations that are not differentiable (and not even necessarily piecewise-differentiable in any reasonable sense). More precisely, we introduce new Lyapunov exponents, for arbitrary transformations, and we consider the

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relation between them and the dimension theory of dynamical systems. Namely, we establish an upper bound for the Hausdorff dimension of a class of invariant measures supported on nonconformal invariant sets (for maps that are not necessarily differentiable), in terms of the new Lyapunov exponents.

Our approach in defining the new exponents is to imitate the derivative as much as possible. An alternative approach could be to replace the derivative cocycle by an appropriate cocycle and then to consider its associated Lyapunov exponent. However, since the transformations that we consider need not even be piecewise-differentiable it is unclear how this approach could be effected.

Relevant departure points for our discussion were the works of Kifer [5] and Barreira [1], where they considered continuous transformations and introduced notions that motivated our own introduction of the new Lyapunov exponents. Namely, Kifer introduced numbers that replace the maximal and minimal values of a Lyapunov exponent. Independently, Barreira introduced numbers that replace the same two values, in the case of repellers of transformations that are not necessarily differentiable. He also used the new exponents to obtain estimates for the Hausdorff dimension of the repellers. Nevertheless, it should be pointed out that it was already clear at that time that in order to obtain sharp dimension estimates it should be necessary to introduce appropriate intermediate values of the Lyapunov exponents. That is, the knowledge of *all* the values is crucial in the dimension theory of nonconformal dynamics. We emphasize that in [5] and [1] the authors consider no intermediate values, but only the maximal and minimal values, contrarily to what is done here.

In the case of differentiable transformations, the new exponents can also be used although one would of course prefer the classical ones. However, since it may not be known a priori whether a given transformation is differentiable, it is crucial to understand how the new Lyapunov exponents relate to the classical ones. Accordingly, we show that for fairly large classes of repellers and of hyperbolic sets the two exponents agree almost everywhere with respect to any finite invariant measure, and thus the two notions can be used indistinctly (almost everywhere). It turns out that the noninvertible version of Oseledets' multiplicative ergodic theorem is crucial for the proof.

The content of the paper is as follows. In Section 2 we recall some notions from the classical theory of Lyapunov exponents, including Oseledets' multiplicative ergodic theorem. In Section 3 we introduce the new exponents and show that they agree almost everywhere with the classical ones for certain classes of transformations. Section 4 describes the relation of the new Lyapunov exponents with the dimension theory of dynamical systems.

2. Lyapunov exponents: classical theory

2.1. **Abstract theory and differentiable transformations.** We begin with the axiomatic definition of Lyapunov exponent. Let V be a linear space. We say that a function $\chi\colon V \to \mathbb{R} \cup \{-\infty\}$ is a *Lyapunov exponent* if:

1. $\chi(\alpha v) = \chi(v)$ for each $v \in V$ and $\alpha \in \mathbb{R} \setminus \{0\}$;
2. $\chi(v + w) \le \max\{\chi(v), \chi(w)\}$ for each $v, w \in V$;

3. $\chi(0) = -\infty$.

We now describe some of the basic properties of Lyapunov exponents (we refer to [2] for full details). Each Lyapunov exponent χ takes at most a number $s \le \dim V$ of distinct real values $\chi_1 < \cdots < \chi_s$ (besides $-\infty$). For each $i = 1, \ldots, s$ we define

$$V_i = \{v \in V : \chi(v) \le \chi_i\}. \tag{1}$$

One can show that each V_i is a linear subspace of V and that

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_s = V. \tag{2}$$

A collection $\{V_i : i = 0, \ldots, s\}$ of linear subspaces of V satisfying (2) is called a *filtration* of V. The filtration defined by the subspaces V_i in (1) is called the *filtration associated* to the Lyapunov exponent χ. We also say that the number

$$k_i = \dim V_i - \dim V_{i-1}$$

is the *multiplicity* of the value χ_i.

Let now $f\colon M \to M$ be a differentiable map on the manifold M. We define the *(forward) Lyapunov exponent* $\lambda^+\colon TM \to \mathbb{R}$ of f by

$$\lambda^+(x, v) = \limsup_{n\to+\infty} \frac{1}{n} \log \|d f_x^n v\|,$$

with the convention that $\log 0 = -\infty$. One can easily show that $T_x M \ni v \mapsto \lambda^+(x, v)$ is indeed a Lyapunov exponent for each $x \in M$, in the sense of the definition above. Therefore, for each $x \in M$ the function $\lambda^+(x, \cdot)$ takes at most a finite number of real values

$$\lambda_1^+(x) < \cdots < \lambda_{s^+(x)}^+(x),$$

for some positive integer $s^+(x) \le \dim M$. Furthermore, the filtration of $T_x M$ associated to $\lambda^+(x, \cdot)$ verifies

$$\{0\} = V_0^+(x) \subset V_1^+(x) \subset \cdots \subset V_{s^+(x)}^+(x) = T_x M,$$

where

$$V_i^+(x) = \{v \in T_x M : \lambda^+(x, v) \le \lambda_i^+(x)\}.$$

The number

$$k_i^+(x) = \dim V_i^+(x) - \dim V_{i-1}^+(x)$$

is the multiplicity of the value $\lambda_i^+(x)$ for each i. We denote by

$$\rho_1^+(x) \le \cdots \le \rho_{\dim M}^+(x)$$

the real values of the Lyapunov exponent $\lambda^+(x, \cdot)$ counted with their multiplicities, i.e., for each i we set

$$\rho^+_{\dim V_{i-1}^+(x)+1}(x) = \cdots = \rho^+_{\dim V_i^+(x)}(x) = \lambda_i^+(x).$$

When the map $f\colon M \to M$ has a differentiable inverse one can also define a Lyapunov exponent for negative time. Namely, the function $\lambda^-\colon TM \to \mathbb{R}$ given by

$$\lambda^-(x, v) = \limsup_{n\to-\infty} \frac{1}{|n|} \log \|d f_x^n v\|$$

is called the *(backward) Lyapunov exponent* of f. Similarly, for each $x \in M$ the function $T_xM \ni v \mapsto \lambda^-(x,v)$ is a Lyapunov exponent, and hence takes at most a finite number of real values

$$\lambda_1^-(x) > \cdots > \lambda_{s^-(x)}^-(x)$$

for some positive integer $s^-(x) \le \dim M$. The filtration of T_xM associated to $\lambda^-(x,\cdot)$ verifies

$$T_xM = V_1^-(x) \supset \cdots \supset V_{s^-(x)}^-(x) \supset V_{s^-(x)+1}^-(x) = \{0\},$$

where

$$V_i^-(x) = \{v \in T_xM : \lambda^-(x,v) \le \lambda_i^-(x)\}.$$

We define the multiplicity of the value $\lambda_i^-(x)$ by

$$k_i^-(x) = \dim V_i^-(x) - \dim V_{i+1}^-(x)$$

for each i, and denote by

$$\rho_1^-(x) \ge \cdots \ge \rho_{\dim M}^-(x)$$

the values of the Lyapunov exponent $\lambda^-(x,\cdot)$ counted with their multiplicities.

2.2. **Regularity and the Multiplicative ergodic theorem.** Using the above structure we can now introduce the fundamental concept of regularity. Let $f\colon M \to M$ be a differentiable map with differentiable inverse. A point $x \in M$ is called *Lyapunov regular* (or simply *regular*) with respect to f if the following holds:

1. $s^+(x) = s^-(x) =: s(x)$;
2. there exists a decomposition
$$T_xM = \bigoplus_{i=1}^{s(x)} H_i(x),$$
such that for each $i = 1, \ldots, s(x)$ we have
$$V_i^+(x) = \bigoplus_{j=1}^{i} H_j(x) \quad \text{and} \quad V_i^-(x) = \bigoplus_{j=i}^{s(x)} H_j(x);$$
3. if $v \in H_i(x) \setminus \{0\}$ then
$$\lim_{n\to\pm\infty} \frac{1}{|n|} \log \|d_xf^n v\| = \lambda_i^+(x) = -\lambda_i^-(x) =: \lambda_i(x),$$
with uniform convergence on $\{v \in H_i(x) : \|v\| = 1\}$;
4. $$\lim_{n\to\pm\infty} \frac{1}{n} \log|\det d_xf^n| = \sum_{i=1}^{s(x)} \lambda_i(x) \dim H_i(x).$$

The notion of regular point is fundamental in the stability theory of linear and nonlinear nonautonomous ordinary differential equations and in the theory of nonuniformly hyperbolic dynamical systems. Namely, the exponential stability of a linear *autonomous* ordinary differential equation $y' = Ay$ is immediate when all the values of the (forward) Lyapunov exponent are negative, and the same happens under sufficiently small perturbations, i.e.,

for $y' = Ay + f(t,y)$, with $f(t,0) = 0$ (and thus keeping zero as a solution of the perturbed equation; one can certainly consider more general situations). On the other hand, an arbitrarily small perturbation

$$y' = A(t)y + f(t,y),$$

with $f(t,0) = 0$, of a linear *nonautonomous* ordinary differential equation $y' = A(t)y$ with all values of the Lyapunov exponent negative may have unstable solutions starting arbitrarily near zero. It turns out that in the presence of regularity (in the sense that there exist regular points) the zero solution of a sufficiently small perturbation of a linear nonautonomous ordinary differential equation with all values of the Lyapunov exponent negative is stable (and in fact asymptotically stable). A similar behavior occurs in the theory of nonuniformly hyperbolic dynamical systems. As such, regularity plays a fundamental role in the stability theory of differential equations and dynamical systems. See [2] for related discussions.

The notion of regular point demands considerably from the structure provided by the Lyapunov exponents λ^+ and λ^-. Therefore, it is crucial to discuss conditions under which there exist regular points. The Multiplicative ergodic theorem of Oseledets [7] provides a surprisingly strong positive answer from the point of view of ergodic theory. We denote by $L^1(M,\mu)$ the set of μ-integrable functions on M. We also write $\log^+ a = \max\{\log a, 0\}$ for each $a \geq 0$.

Theorem 1 (Multiplicative ergodic theorem). *If $f\colon M \to M$ is a differentiable map with differentiable inverse and μ is a finite f-invariant measure on M such that $\log^+ \|df\|$, $\log^+ \|df^{-1}\| \in L^1(M,\mu)$, then μ-almost every $x \in M$ is regular.*

Theorem 1 says that from the point of view of ergodic theory regularity is a typical property. In particular, if $f\colon M \to M$ is a C^1 diffeomorphism of a compact manifold, then the set of regular points has full measure with respect to any finite f-invariant measure on M.

When f is not necessarily invertible we still have the following result.

Theorem 2. *If $f\colon M \to M$ is a differentiable map and μ is a finite f-invariant measure on M such that $\log^+ \|df\| \in L^1(M,\mu)$, then for μ-almost every $x \in M$ the following properties hold:*

1. *for $i = 1, \ldots, s^+(x)$ and $v \in V_i^+(x) \setminus V_{i-1}^+(x)$ we have*
$$\lim_{n\to+\infty} \frac{1}{n} \log \|df_x^n v\| = \lambda_i^+(x), \tag{3}$$
with uniform convergence in v on each subspace $F \subset V_i^+(x)$ such that $F \cap V_{i-1}^+(x) = \{0\}$;
2. $$\lim_{n\to+\infty} \frac{1}{n} \log|\det df_x^n| = \sum_{i=1}^{s^+(x)} k_i^+(x)\lambda_i^+(x). \tag{4}$$

In particular, the uniform convergence in Theorem 2 is essential in Section 3. The statement in Theorem 2 is due to Oseledets, even though, strictly speaking, he only discussed the property of uniform convergence in the invertible case (that in our setup corresponds to the situation in Theorem 1).

Furthermore, to our best knowledge, the property of uniform convergence was only formulated explicitly in the appendix to [4]. Nevertheless, the statement can indeed be recovered by essentially following arguments of Oseledets. For completeness—and since the uniformity is essential in Section 3—we give a simple proof of this property. For a proof of the remaining statements in Theorem 2 see for example [2].

Proof of the uniform convergence in Theorem 2. Let X be the full measure set composed of the points $x \in M$ for which there exist the limits in (3) and (4), *without assuming the uniform convergence in* (3). Let $x \in X$. We need to show that

$$\lim_{n\to+\infty}\frac{1}{n}\log\inf_{v\in C}\|df_x^n v\| = \lim_{n\to+\infty}\frac{1}{n}\log\sup_{v\in C}\|df_x^n v\| = \lambda_i^+(x),$$

where $C = \{v \in F : \|v\| = 1\}$.

Consider an orthonormal basis $e_1, \ldots, e_m$ of F, and for each $n \in \mathbb{N}$ let

$$u_n = \sum_{j=1}^m c_{n,j} e_j \in C$$

be a vector at which $v \mapsto \|df_x^n v\|$ attains its minimum. We choose an integer $j(n) \in \{1, \ldots, m\}$ such that $|c_{n,j(n)}| = \max_j |c_{n,j}|$. Since $\sum_{j=1}^m {c_{n,j}}^2 = 1$, we have

$$|c_{n,j(n)}| \ge 1/\sqrt{m}.$$

We denote by $\rho_{n,j}$ and $\varphi_{n,j}$ respectively the distance and the angle between $df_x^n e_j$ and $df_x^n \operatorname{span}\{e_i : i \ne j\}$. Note that $\rho_{n,j} = \|df_x^n e_j\| \sin\varphi_{n,j}$. We have

$$df_x^n u_n = c_{n,j(n)}\left(df_x^n e_{j(n)} + \sum_{j\ne j(n)} \frac{c_{n,j}}{c_{n,j(n)}} df_x^n e_j\right),$$

and hence

$$\|df_x^n u_n\| \ge |c_{n,j(n)}|\rho_{n,j(n)} \ge \frac{1}{\sqrt{m}}\|df_x^n e_{j(n)}\| \sin\varphi_{n,j(n)}.$$

Using the existence of the limits in (3) and (4) it can be shown that

$$\lim_{n\to+\infty}\frac{1}{n}\log|\sin\varphi_{n,j}| = 0$$

(and this does not require the uniform convergence in (3); see Theorem 1.5.1 in [2]). Since $j(n)$ can only take a finite number of values, we obtain

$$\begin{aligned}
&\liminf_{n\to+\infty}\frac{1}{n}\log\|df_x^n u_n\| \\
&\ge \liminf_{n\to+\infty}\frac{1}{n}\log\|df_x^n e_{j(n)}\| + \liminf_{n\to+\infty}\frac{1}{n}\log|\sin\varphi_{n,j(n)}| \qquad (5)\\
&\ge \min_j \liminf_{n\to+\infty}\frac{1}{n}\log\|df_x^n e_j\| + \min_j \liminf_{n\to+\infty}\frac{1}{n}\log|\sin\varphi_{n,j}| = \lambda_i^+(x),
\end{aligned}$$

where the minima are taken over all $j \in \{1, \ldots, m\}$.

For each $n \in \mathbb{N}$, choose now a vector

$$v_n = \sum_{j=1}^m d_{n,j} e_j \in C$$

at which $v \mapsto \|df_x^n v\|$ attains its maximum. We have

$$\|df_x^n v_n\| \le \sum_{j=1}^m |d_{n,j}| \cdot \|df_x^n e_j\| \le \sum_{j=1}^m \|df_x^n e_j\|,$$

and hence

$$\limsup_{n\to+\infty} \frac{1}{n} \log\|df_x^n v_n\| \le \lambda_i^+(x). \tag{6}$$

The uniform convergence is the combination of (5) and (6). □

3. Lyapunov exponents for continuous transformations

We introduce in this section a version of Lyapunov exponents for maps that are not necessarily differentiable. Following the abstract theory briefly presented in Section 2.1 it is reasonable to consider as many values of the exponent (counted with their multiplicities) as the dimension of the whole space. The basic idea behind the definition is to imitate the derivative as much as possible.

3.1. **The new Lyapunov exponents.** Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a transformation on $\mathbb{R}^m$. We emphasize that f is not necessarily differentiable.

Let d be the metric on $\mathbb{R}^m$. For each $x \in \mathbb{R}^m$ and $k \in \{1, \ldots, m\}$ we define the number

$$\Lambda_k^+(x) = \inf_{L\in L_{x,k}} \lim_{\delta\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log \sup_{y\in C_x(\delta,n)\cap L} \frac{d(f^n x, f^n y)}{d(x,y)}, \tag{7}$$

where $L_{x,k}$ denotes the family of sets of the form $x + F$ for some subspace $F \subset \mathbb{R}^m$ of dimension k, and

$$C_x(\delta, n) = \{y \in B_x(\delta, n) \setminus \{x\} : [f^j x, f^j y] \subset f^j B_x(\delta, n) \text{ for } j \in \{0, \ldots, n\}\};$$

here

$$B_x(\delta, n) = \{y \in \mathbb{R}^m : d(f^j x, f^j y) < \delta \text{ for } j \in \{0, \ldots, n\}\}, \tag{8}$$

and $[v, w] \subset \mathbb{R}^m$ denotes the line segment between the points v and w. We refer to the numbers

$$\Lambda_1^+(x) \le \Lambda_2^+(x) \le \cdots \le \Lambda_m^+(x)$$

as the *Lyapunov exponents* of f at the point x (note that there is no danger of confusion with the classical Lyapunov exponents). These numbers play the role of the Lyapunov exponents in the case of maps that are not differentiable. We observe that the value of each $\Lambda_k^+(x)$ does not change when we replace d by an equivalent metric in (7) and (8).

It would also be interesting to define an appropriate Lyapunov exponent (in the sense of the abstract theory in Section 2.1) for which the numbers that we introduce would be the values of that Lyapunov exponent (again in the sense of Section 2.1). However, this approach seems to be of rather formal nature in our context. Nevertheless, it has been applied with success in other contexts, such as in the study of a pendulum with dry friction in [6] (in this case the system is piecewise-differentiable, and thus the situation is unrelated to ours).

3.2. **Relation with the classical Lyapunov exponents.** We now consider a differentiable transformation $f\colon \mathbb{R}^m \to \mathbb{R}^m$. For each $x \in \mathbb{R}^m$ and $k \in \{1, \ldots, m\}$ we define the number

$$c_k^+(x) = \inf_{\dim F = k} \limsup_{n\to+\infty} \frac{1}{n} \log \sup_{v\in F\setminus\{0\}} \frac{\|d f_x^n v\|}{\|v\|},$$

where the infimum is taken over all subspaces $F \subset \mathbb{R}^m$ of dimension k. Clearly

$$c_1^+(x) \le c_2^+(x) \le \cdots \le c_m^+(x).$$

We now relate these numbers with the values $\rho_k^+(x)$ of the classical (forward) Lyapunov exponent at x, i.e., the values of the Lyapunov exponent $\lambda^+(x, \cdot)$ counted with their multiplicities (see Section 2.1).

Theorem 3. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a differentiable map and μ a finite f-invariant measure on $\mathbb{R}^m$ such that $\log^+ \|df\| \in L^1(\mathbb{R}^m, \mu)$. Then, for μ-almost every $x \in \mathbb{R}^m$ and each $k = 1, \ldots, m$ we have*

$$\rho_k^+(x) = c_k^+(x) = \lim_{n\to+\infty} \frac{1}{n} \log \sup_{v\in G\setminus\{0\}} \frac{\|d f_x^n v\|}{\|v\|} = \lambda_i^+(x),$$

where G is any subspace of dimension $k > \dim V_{i-1}^+(x)$ such that $G \subset V_i^+(x)$ for some i.

Proof. By Theorem 2, for μ-almost every $x \in \mathbb{R}^m$ and each $i = 1, \ldots, s^+(x)$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \log \frac{\|d f_x^n v\|}{\|v\|} = \lambda_i^+(x)$$

whenever $v \in V_i^+(x) \setminus V_{i-1}^+(x)$, with uniform convergence on each subspace $F \subset V_i^+(x)$ such that $F \cap V_{i-1}^+(x) = \{0\}$. Let $x \in \mathbb{R}^m$ be one of these points. Given $i \in \{1, \ldots, s^+(x)\}$, let $G \subset V_i^+(x)$ be a subspace of dimension $k > \dim V_{i-1}^+(x)$. We first show that

$$\lim_{n\to+\infty} \frac{1}{n} \log \sup_{v\in G\setminus\{0\}} \frac{\|d f_x^n v\|}{\|v\|} = \lambda_i^+(x) = \rho_k^+(x). \tag{9}$$

This is clear when $i = 1$ (simply take $F = G$ in Theorem 2). We proceed by induction on i. We first observe that

$$\lim_{n\to+\infty} \frac{1}{n} \log \sup_{v\in G\setminus\{0\}} \frac{\|d f_x^n v\|}{\|v\|} \ge \lim_{n\to+\infty} \frac{1}{n} \log \frac{\|d f_x^n w\|}{\|w\|} = \lambda_i^+(x) = \rho_k^+(x)$$

whenever $w \in G \setminus V_{i-1}^+(x)$.

We now establish the reverse inequality. Let $v_n \in G$ be a vector with $\|v_n\| = 1$ at which the supremum in (9) is attained. We write $v_n = a_n + b_n$, where $a_n \in V_i^+(x) \cap (V_{i-1}^+(x))^\perp$ and $b_n \in V_{i-1}^+(x)$. Clearly $\|a_n\| \le 1$ and $\|b_n\| \le 1$, and thus

$$\begin{aligned} \|d f_x^n v_n\| &\le \|d f_x^n a_n\| + \|d f_x^n b_n\| \\ &\le \sup_{v\in V_i^+(x)\cap(V_{i-1}^+(x))^\perp\setminus\{0\}} \frac{\|d f_x^n v\|}{\|v\|} + \sup_{v\in V_{i-1}^+(x)\setminus\{0\}} \frac{\|d f_x^n v\|}{\|v\|}. \end{aligned} \tag{10}$$

By the induction hypothesis,

$$\lim_{n\to+\infty}\frac{1}{n}\log\sup_{v\in V_{i-1}^+(x)\setminus\{0\}}\frac{\|df_x^n v\|}{\|v\|}=\lambda_{i-1}^+(x)<\lambda_i^+(x).$$

On the other hand, taking $F=V_i^+(x)\cap(V_{i-1}^+(x))^\perp$ in Theorem 2 we obtain

$$\lim_{n\to+\infty}\frac{1}{n}\log\sup_{v\in V_i^+(x)\cap(V_{i-1}^+(x))^\perp\setminus\{0\}}\frac{\|df_x^n v\|}{\|v\|}=\lambda_i^+(x)=\rho_k^+(x).$$

It now follows from (10) that

$$\lim_{n\to+\infty}\frac{1}{n}\log\sup_{v\in G\setminus\{0\}}\frac{\|df_x^n v\|}{\|v\|}\le\lambda_i^+(x)=\rho_k^+(x).$$

This establishes the identities in (9).

For every subspace $G\subset V_i^+(x)$ of dimension $k>\dim V_{i-1}^+(x)$, it follows from (9) that

$$c_k^+(x)\le\lim_{n\to+\infty}\frac{1}{n}\log\sup_{v\in G\setminus\{0\}}\frac{\|df_x^n v\|}{\|v\|}=\rho_k^+(x).$$

On the other hand, for each subspace F of dimension $k>\dim V_{i-1}^+(x)$ there exists $v_F\in F\setminus V_{i-1}^+(x)$ such that

$$\lim_{n\to+\infty}\frac{1}{n}\log\frac{\|df_x^n v_F\|}{\|v_F\|}\ge\lambda_i^+(x)=\rho_k^+(x),$$

and thus

$$c_k^+(x)\ge\inf_{\dim F=k}\lim_{n\to+\infty}\frac{1}{n}\log\frac{\|df_x^n v_F\|}{\|v_F\|}\ge\lambda_i^+(x)=\rho_k^+(x).$$

This establishes the desired result. □

We now start discussing the relation between the new exponents $\Lambda_k^+(x)$ and the classical Lyapunov exponents.

Proposition 4. *If $f\colon\mathbb{R}^m\to\mathbb{R}^m$ is a differentiable map then $\Lambda_k^+(x)\ge c_k^+(x)$ for each $x\in\mathbb{R}^m$ and $k=1,\dots,m$.*

Proof. Let $L=x+F\in L_{x,k}$. For each $n\in\mathbb{N}$ we choose $v_{n,F}\in F$ (possibly depending on x) with $\|v_{n,F}\|=1$ such that

$$\sup_{v\in F\setminus\{0\}}\frac{\|df_x^n v\|}{\|v\|}=\|df_x^n v_{n,F}\|.$$

The differentiability of f ensures that

$$\lim_{\varepsilon\to0}\frac{d(f^n x,f^n(x+\varepsilon v_{n,F}))}{d(x,x+\varepsilon v_{n,F})}=\|df_x^n v_{n,F}\|.$$

On the other hand, for each sufficiently small $\varepsilon>0$ we have $x+\varepsilon v_{n,F}\in C_x(\delta,n)$, and thus

$$\sup_{y\in C_x(\delta,n)\cap L}\frac{d(f^n x,f^n y)}{d(x,y)}\ge\frac{d(f^n x,f^n(x+\varepsilon v_{n,F}))}{d(x,x+\varepsilon v_{n,F})}.$$

Letting $\varepsilon \to 0$ yields

$$\sup_{y\in C_x(\delta,n)\cap L} \frac{d(f^n x, f^n y)}{d(x,y)} \ge \|df_x^n v_{n,F}\|,$$

and hence

$$\Lambda_k^+(x) \ge \inf_{\dim F=k} \limsup_{n\to+\infty} \frac{1}{n} \log \|df_x^n v_{n,F}\| = c_k^+(x).$$

This completes the proof. □

Before proceeding we recall the notion of repeller. Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a differentiable map and $J \subset \mathbb{R}^m$ a compact f-invariant set (i.e., such that $f^{-1}J = J$). We say that f is *expanding* on J and that J is a *repeller* of f if there exist constants $c > 0$ and $a > 1$ such that

$$\|df_x^n v\| \ge ca^n \|v\|$$

for each $x \in J$, $v \in \mathbb{R}^m$ and $n \in \mathbb{N}$. Note that in this case f is a diffeomorphism on some neighborhood of each element of J.

Let now $\alpha \in (0,1]$. We say that a differentiable map $f\colon \mathbb{R}^m \to \mathbb{R}^m$ has *α-bunched derivative* on the set J if df_x is invertible and

$$\|(df_x)^{-1}\|^{1+\alpha} \|df_x\| < 1$$

for every $x \in J$. This property was introduced in [1]. The simplest nontrivial examples of expanding maps with bunched derivative are the following.

Example 1. Consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $fx = Ax$, where

$$A = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

for some p, $q \in \mathbb{Z}$ such that $|p| > |q| > 1$. One can easily choose p and q so that f has α-bunched derivative on $\mathbb{R}^2$. In fact,

$$\|(df_x)^{-1}\|^{1+\alpha} \|df_x\| = |q|^{-(1+\alpha)} |p|,$$

and thus, when $|p| < |q|^{1+\alpha}$ the map f has α-bunched derivative.

We can also consider similar examples in $\mathbb{R}^m$. Consider the map $f\colon \mathbb{R}^m \to \mathbb{R}^m$ defined by $fx = Ax$, for some invertible matrix A with distinct eigenvalues, all of which with absolute value bigger than one. We have

$$\|(df_x)^{-1}\|^{1+\alpha} \|df_x\| = (\min_i |\lambda_i|)^{-1-\alpha} \max_i |\lambda_i|,$$

and choosing the numbers λ_i in such a way that $\max_i |\lambda_i| < (\min_i |\lambda_i|)^{-1-\alpha}$, the map f has α-bunched derivative on $\mathbb{R}^m$.

We note that any sufficiently small C^1 perturbation of a C^1 map with α-bunched derivative still possesses this property (with the same α).

We now continue the study of the relation between the new Lyapunov exponents and the classical ones. We want to show that the two notions agree in the case of repellers with bunched derivative. The following is the main step in our approach.

Theorem 5. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a $C^{1+\alpha}$ transformation with an f-invariant compact repeller J on which f has α-bunched derivative. Then $\Lambda_k^+(x) = c_k^+(x)$ for each $x \in J$ and $k = 1, \ldots, m$.*

Proof. By Proposition 4 it remains to show that $\Lambda_k^+(x) \le c_k^+(x)$ for each $x \in J$. Choose $\delta = \delta(x) > 0$ such that f is a diffeomorphism on the ball $B_x(\delta)$ of radius δ centered at x. For each $n \in \mathbb{N}$ we have $B_x(\delta, n) \subset B_x(\delta)$ and hence f is also a local diffeomorphism on $B_x(\delta, n)$.

Consider $y \in C_x(\delta, n+1)$ and $j \in \{0, \dots, n\}$. We have

$$\begin{aligned}
&df_y^{j+1}(df_x^{j+1})^{-1}\\
&= df_{fy}^j df_y (df_x)^{-1} (df_{fx}^j)^{-1}\\
&= df_{fy}^j (df_{fx}^j)^{-1} + df_{fy}^j df_y (df_x)^{-1} (df_{fx}^j)^{-1}\\
&\quad - df_{fy}^j (df_{fx}^j)^{-1}\\
&= df_{fy}^j (df_{fx}^j)^{-1} + df_{fy}^j df_y (df_x)^{-1} (df_{fy}^j)^{-1} df_{fy}^j (df_{fx}^j)^{-1}\\
&\quad - df_{fy}^j (df_{fy}^j)^{-1} df_{fy}^j (df_{fx}^j)^{-1}\\
&= \left[I + df_{fy}^j df_y (df_x)^{-1} (df_{fy}^j)^{-1} - df_{fy}^j (df_{fy}^j)^{-1}\right] df_{fy}^j (df_{fx}^j)^{-1}\\
&= \left[I + df_{fy}^j (df_y (df_x^{-1}) - I)(df_{fy}^j)^{-1}\right] df_{fy}^j (df_{fx}^j)^{-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\|df_y^{j+1}(df_x^{j+1})^{-1}\|}{\|df_{fy}^j (df_{fx}^j)^{-1}\|} &\le 1 + \|df_{fy}^j\| \cdot \|(df_y (df_x)^{-1} - I)(df_{fy}^j)^{-1}\|\\
&\le 1 + C_1 \|df_{fy}^j\| \cdot \|df_y - df_x\| \cdot \|(df_{fy}^j)^{-1}\|\\
&\le 1 + C_1 C_2 \|df_{fy}^j\| \cdot \|y - x\|^\alpha \|(df_{fy}^j)^{-1}\|,
\end{aligned} \tag{11}$$

where

$$C_1 = \max\{\|(df_v)^{-1}\| : v \in J\},$$

and $C_2 > 0$ is chosen in such a way that

$$\|df_v - df_w\| \le C_2 \|v - w\|^\alpha$$

for every v, $w \in \mathbb{R}^m$ (recall that f is of class $C^{1+\alpha}$).

We now estimate each of the terms in the right-hand side of (11), starting with $\|y - x\|^\alpha$. Let h_j be the local inverse of f^j on $B_{f^j x}(\delta)$. We have

$$\|y - x\| = \|h_j(f^j y) - h_j(f^j x)\| \le \|d(h_j)_z\| \cdot \|f^j y - f^j x\|, \tag{12}$$

where z is some point in the line segment between $f^j y$ and $f^j x$, and thus also in $f^j B_x(\delta, n+1)$, by the definition of $C_x(\delta, n+1)$. Therefore

$$f^\ell h_j z \in f^\ell B_x(\delta, n+1) \subset B_{f^\ell x}(\delta) \tag{13}$$

for $\ell = 0, \dots, j$.

Since the derivative of f is α-bunched on J and J is compact, we can choose $\lambda < 1$ sufficiently large and δ sufficiently small so that

$$\|(df_z)^{-1}\|^{1+\alpha} \|df_z\| < \lambda$$

for every z in the δ-neighborhood J_δ of J. Let now $\beta > 0$ be such that $e^{\alpha\beta}\lambda < 1$. Eventually making again δ sufficiently small, we can assume that if v, $w \in J_\delta$ and $d(v, w) < 2\delta$, then

$$\left|\log \|(df_v)^{-1}\| - \log \|(df_w)^{-1}\|\right| \le \beta$$

(note that J is compact and $v \mapsto \log \|(df_v)^{-1}\|$ is continuous). It follows from (13) that

$$\|(df_{f^\ell h_j z})^{-1}\| \le e^\beta \|(df_{f^\ell y})^{-1}\| \tag{14}$$

for $\ell = 0, \ldots, n$.

We now return to (12). Since $d(h_j)_z = (df^j_{h_j z})^{-1}$, it follows from (14) that

$$\|d(h_j)_z\| = \|(df^j_{h_j z})^{-1}\| \le \prod_{\ell=0}^{j-1} \|(df_{f^\ell h_j z})^{-1}\| \le C_3 e^{\beta j} \prod_{\ell=1}^{j} \|(df_{f^\ell y})^{-1}\|,$$

where

$$C_3 = \sup_w \|(df_w)^{-1}\| / \inf_w \|(df_w)^{-1}\|$$

with the supremum and infimum taken over all $w \in J_\delta$. It follows from (12) that

$$\|y - x\|^\alpha \le \delta^\alpha \|d(h_j)_z\|^\alpha \le {C_3}^\alpha \delta^\alpha e^{\alpha\beta j} \prod_{\ell=1}^{j} \|(df_{f^\ell y})^{-1}\|^\alpha. \tag{15}$$

The remaining terms in the right-hand side of (11) are estimated by

$$\|df^j_{fy}\| \|(df^j_{fy})^{-1}\| \le \prod_{\ell=1}^{j} (\|df_{f^\ell y}\| \cdot \|(df_{f^\ell y})^{-1}\|). \tag{16}$$

We conclude from (15) and (16) that

$$\|y - x\|^\alpha \|(df^j_{fy})^{-1}\| \|df^j_{fy}\| \le {C_3}^\alpha \delta^\alpha e^{\alpha\beta j} \prod_{\ell=1}^{j} \left(\|(df_{f^\ell y})^{-1}\|^{1+\alpha} \|df_{f^\ell y}\| \right).$$

Therefore

$$\|y - x\|^\alpha \|(df^j_{fy})^{-1}\| \|df^j_{fy}\| \le {C_3}^\alpha \delta^\alpha (e^{\alpha\beta}\lambda)^j.$$

It follows from (11) that for each $x \in J$, $n \in \mathbb{N}$, $y \in C_x(\delta, n+1)$, and $j \in \{0, \ldots, n\}$ we have

$$\|df^{j+1}_y (df^{j+1}_x)^{-1}\| \le \|df^j_{fy} (df^j_{fx})^{-1}\| (1 + C\gamma^j),$$

where $C = C_1 C_2 {C_3}^\alpha \delta^\alpha$. We conclude that

$$\|df^{n+1}_y (df^{n+1}_x)^{-1}\| \le \prod_{j=1}^{n} (1 + C\gamma^j) < \prod_{j=1}^{\infty} (1 + C\gamma^j) = \tau.$$

Consider now a subspace F of $\mathbb{R}^m$. For each linear map $A\colon \mathbb{R}^m \to \mathbb{R}^m$ we write

$$\|A\|_F = \sup_{v \in F \setminus \{0\}} \frac{\|Av\|}{\|v\|}.$$

Note that given another linear transformation $B\colon \mathbb{R}^m \to \mathbb{R}^m$ we have

$$\|BA\|_F = \sup_{v \in F \setminus \{0\}} \frac{\|BAv\|}{\|v\|} \le \|B\| \cdot \|A\|_F.$$

Setting $A = df^n_x$ and $B = df^n_y (df^n_x)^{-1}$ we conclude that

$$\frac{\|df^n_y\|_F}{\|df^n_x\|_F} \le \|df^n_y (df^n_x)^{-1}\| \le \tau$$

whenever $x \in J$, $n \in \mathbb{N}$, and $y \in C_x(\delta, n)$. Assume now that $w = y - x \in F$. Then

$$d(f^n x, f^n y) \leq \int_0^1 \|df^n_{x+tw}\|_F dt$$
$$\leq \sup_{z \in C_x(\delta,n)\cap(x+F)} \|df^n_z\|_F d(x,y) \leq \tau \|df^n_x\|_F d(x,y).$$

It follows that

$$\limsup_{n\to+\infty} \frac{1}{n} \log \sup_{y\in C_x(\delta,n)\cap(x+F)} \frac{d(f^n x, f^n y)}{d(x,y)} \leq \limsup_{n\to+\infty} \frac{1}{n} \log \|df^n_x\|_F,$$

and hence

$$\Lambda^+_k(x) \leq \inf_{\dim F = k} \limsup_{n\to+\infty} \frac{1}{n} \log \|df^n_x\|_F = c^+_k(x).$$

This completes the proof. □

The following is now immediate.

Corollary 6. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a $C^{1+\alpha}$ transformation with an f-invariant compact repeller J on which f has α-bunched derivative, and let μ be a finite f-invariant measure on J. Then*

$$\Lambda^+_k(x) = c^+_k(x) = \rho^+_k(x)$$

for μ-almost every $x \in J$ and each $k = 1, \ldots, m$.

Proof. The result follows immediately from Theorems 3 and 5. □

A version of Theorem 5 was given by Kifer in [5] in the case of the maximal and minimal values introduced in that paper (and thus not including any of the intermediate values introduced here). We note that Kifer considers arbitrary differentiable flows (and semiflows).

3.3. **Invertible transformations.** We now assume that $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is invertible. In a similar manner to that in Section 3.1, for each $x \in \mathbb{R}^m$ and $k \in \{1, \ldots, m\}$ we define the number

$$\Lambda^-_k(x) = \inf_{L \in L_{x,m-k+1}} \lim_{\delta\to 0} \limsup_{n\to-\infty} \frac{1}{|n|} \log \sup_{y \in C_x(\delta,n)\cap L} \frac{d(f^n x, f^n y)}{d(x,y)},$$

where

$$C_x(\delta, n) = \{y \in B_x(\delta, n) \setminus \{x\} : [f^j x, f^j y] \subset f^j B_x(\delta, n) \text{ for } j \in \{n, \ldots, 0\}\}$$

and

$$B_x(\delta, n) = \{y \in \mathbb{R}^m : d(f^j x, f^j y) < \delta \text{ for each } j \in \{n, \ldots, 0\}\}.$$

We refer to the numbers $\Lambda^-_k(x)$ as the *(backward) Lyapunov exponents* of f at the point x. For each $x \in \mathbb{R}^m$ and $k \in \{1, \ldots, m\}$ we define the number

$$c^-_k(x) = \inf_{\dim F = m-k+1} \limsup_{n\to-\infty} \frac{1}{|n|} \log \sup_{v \in F\setminus\{0\}} \frac{\|df^n_x v\|}{\|v\|}.$$

Clearly

$$\Lambda^-_1(x) \geq \Lambda^-_2(x) \geq \cdots \geq \Lambda^-_m(x) \quad \text{and} \quad c^-_1(x) \geq c^-_2(x) \geq \cdots \geq c^-_m(x).$$

We can also obtain similar results to those above in the case of negative time. More precisely, altering the proofs in a straightforward manner, we obtain the following.

Theorem 7. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a map with differentiable inverse and μ a finite f-invariant measure on $\mathbb{R}^m$ such that $\log^+ \|df^{-1}\| \in L^1(\mathbb{R}^m, \mu)$. Then, for μ-almost every $x \in \mathbb{R}^m$ and each $k = 1, \ldots, m$ we have*

$$\rho_k^-(x) = c_k^-(x) = \limsup_{n\to-\infty} \frac{1}{|n|} \log \sup_{v\in G\setminus\{0\}} \frac{\|df_x^n v\|}{\|v\|} = \lambda_i^-(x),$$

where G is any subspace of dimension $k > \dim V_i^-(x)$ such that $G \subset V_{i+1}^-(x)$ for some i.

Proposition 8. *If $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is a differentiable map with differentiable inverse then*

$$\Lambda_k^+(x) \geq c_k^+(x) \quad \text{and} \quad \Lambda_k^-(x) \geq c_k^-(x)$$

for each $x \in \mathbb{R}^m$ and $k = 1, \ldots, m$.

We can also change the proof of Theorem 5 and establish an appropriate version in the case of hyperbolic sets. Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a diffeomorphism. We recall that a compact f-invariant set $\Lambda \subset \mathbb{R}^m$ is a *hyperbolic set* for f if there exist a splitting $\mathbb{R}^m = E^s(x) \oplus E^u(x)$ varying continuously with $x \in \Lambda$, and constants $c > 0$ and $\lambda \in (0,1)$ such that for each $x \in \Lambda$:

1. $df_x(E_x^u) = E_{fx}^u$ and $df_x(E_x^s) = E_{fx}^s$;
2. $\|df_x^n v\| \leq c\lambda^n \|v\|$ whenever $v \in E_x^s$ and $n \in \mathbb{N}$;
3. $\|df_x^{-n} v\| \leq c\lambda^n \|v\|$ whenever $v \in E_x^u$ and $n \in \mathbb{N}$.

We write for simplicity $A_s(x) = df_x|E^s(x)$ and $A_u(x) = df_x|E^u(x)$.

Theorem 9. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a $C^{1+\alpha}$ diffeomorphism with an f-invariant compact hyperbolic set Λ such that*

$$\|A_s(x)\|^{1+\alpha} \|(A_s(x))^{-1}\| < 1 \quad \text{and} \quad \|(A_u(x))^{-1}\|^{1+\alpha} \|A_u(x)\| < 1 \tag{17}$$

for every $x \in \Lambda$. Then

$$\Lambda_k^+(x) = c_k^+(x) = -\Lambda_k^-(x) = -c_k^-(x)$$

for each $x \in \Lambda$ and $k = 1, \ldots, m$.

Proof. Since the distributions E^s and E^u are df-invariant, using (17) one can reproduce arguments in the proof of Theorem 5 to conclude that there exists $\tau > 0$ such that for each $x \in \Lambda$ and $y \in C_x(\delta, n)$ we have

$$\|df_y^n | E^s(x)\| \leq \tau \|df_x^n | E^s(x)\| \tag{18}$$

whenever $n < 0$, and

$$\|df_y^n | E^u(x)\| \leq \tau \|df_x^n | E^u(x)\| \tag{19}$$

whenever $n > 0$.

Observe now that for each k the numbers $c_k^+(x)$ and $c_k^-(x)$ are nonzero and $c_k^+(x) = -c_k^-(x) =: c_k(x)$. This is a consequence of the uniform estimates in the definition of a hyperbolic set. Furthermore, again since the distributions E^s and E^u are df-invariant, one can easily show that each $c_k(x)$ can be

computed by solely considering subspaces F inside $E^s(x)$ or $E^u(x)$. More precisely, if $c_k(x) < 0$ then

$$\begin{aligned} c_k(x) &= \inf_F \limsup_{n\to+\infty} \frac{1}{n} \log \sup_{v\in F\setminus\{0\}} \frac{\|df_x^n v\|}{\|v\|} \\ &= -\inf_G \limsup_{n\to-\infty} \frac{1}{|n|} \log \sup_{v\in G\setminus\{0\}} \frac{\|df_x^n v\|}{\|v\|}, \end{aligned}$$

where the infimum is taken over all subspaces F, $G \subset E^s(x)$ respectively of dimensions k and $\dim E^s(x) - k + 1$. Similarly, if $c_k(x) > 0$ then

$$\begin{aligned} c_k(x) &= \inf_F \limsup_{n\to+\infty} \frac{1}{n} \log \sup_{v\in F\setminus\{0\}} \frac{\|df_x^n v\|}{\|v\|} \\ &= -\inf_G \limsup_{n\to-\infty} \frac{1}{|n|} \log \sup_{v\in G\setminus\{0\}} \frac{\|df_x^n v\|}{\|v\|}, \end{aligned}$$

where the infimum is taken over all subspaces F, $G \subset E^s(x)$ respectively of dimensions $k - \dim E^s(x)$ and $m - k + 1$. Using these formulas together with the inequalities in (18) and (19) one can reproduce arguments in the proof of Theorem 5 to show that

$$\Lambda_k^+(x) \le c_k^+(x) \quad \text{and} \quad \Lambda_k^-(x) \le c_k^-(x).$$

The desired result follows now immediately from Proposition 8. $\square$

The following is now an immediate consequence.

Corollary 10. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a $C^{1+\alpha}$ diffeomorphism with an f-invariant compact hyperbolic set Λ satisfying* (17) *for every $x \in \Lambda$, and μ a finite f-invariant measure on Λ. Then*

$$\Lambda_k^+(x) = \rho_k^+(x) = c_k^+(x) = -\Lambda_k^-(x) = -\rho_k^-(x) = -c_k^-(x)$$

for μ-almost every $x \in \Lambda$ and each $k = 1, \ldots, m$.

We note that the new notion of Lyapunov exponent can easily be extended to flows (and semiflows), and analogous proofs allow us to obtain corresponding versions of the previous results. Following our approach, we can also consider Lyapunov exponents for maps defined in smooth manifolds with the help of the exponential map.

4. Relation with dimension theory

We illustrate here how the Lyapunov exponents introduced in Section 3 are related with dimension theory. Instead of describing more general situations we consider on purpose a particular class of transformations for which our results are still nontrivial. More precisely, we consider invariant sets of a class of "expanding" transformations that need not be differentiable.

4.1. **Basic notions.** We denote by $\dim_H F$ the Hausdorff dimension of a set $F \subset \mathbb{R}^m$. Given a finite measure μ on $\mathbb{R}^m$ we define the *Hausdorff dimension of μ* by

$$\dim_H \mu = \inf\{\dim_H Z : \mu(\mathbb{R}^m \setminus Z) = 0\}.$$

We also need the alternative approach of Douady and Oesterlé in [3] to the Hausdorff dimension. Given an ellipsoid $E \subset \mathbb{R}^m$ with semiaxes $a_m(E) \geq \cdots \geq a_1(E)$, for each $t \in [0,m]$ we consider the number

$$w_t(E) = a_m(E) \cdots a_{m-\lfloor t\rfloor+1}(E)(a_{m-\lfloor t\rfloor}(E))^{t-\lfloor t\rfloor}.$$

For each set $A \subset \mathbb{R}^m$ let

$$\mu(A,t) = \lim_{\varepsilon\to 0} \inf_{\mathcal{U}} \sum_{E\in\mathcal{U}} w_t(E),$$

where the infimum is taken over all finite or countable covers of A by ellipsoids E such that $w_d(E)^{1/t} \leq \varepsilon$. The following result in [3, Proposition 1] provides an alternative definition for the Hausdorff dimension.

Proposition 11. $\dim_H A = \inf\{t : \mu(A,t) = 0\} = \sup\{t : \mu(A,t) = +\infty\}$.

We now briefly recall the nonadditive version of the topological pressure introduced in [1]. Let $f\colon X \to X$ be a continuous map of a compact metric space, and $\mathcal{U}$ a finite open cover of X. Given $U = (U_1,\ldots,U_n) \in \mathcal{U}^n$ we define the open set

$$X(U) = \bigcap_{k=1}^{n} f^{-k+1}U_k.$$

Consider a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ of functions $\varphi_n\colon X \to \mathbb{R}$. For each $n \in \mathbb{N}$ we define

$$\gamma_n(\Phi,\mathcal{U}) = \sup\{|\varphi_n(x) - \varphi_n(y)| : x,\, y \in X(U) \text{ for some } U \in \mathcal{U}^n\}.$$

Let $\operatorname{diam}\mathcal{U}$ be the diameter of the cover $\mathcal{U}$. We assume that

$$\limsup_{\operatorname{diam}\mathcal{U}\to 0}\ \limsup_{n\to\infty} \frac{\gamma_n(\Phi,\mathcal{U})}{n} = 0. \tag{20}$$

Given $U \in \mathcal{U}^n$ set

$$\Phi(U) = \begin{cases} \sup_{X(U)} \varphi_n & \text{if } X(U) \neq \varnothing \\ -\infty & \text{if } X(U) = \varnothing \end{cases}.$$

For each $Z \subset X$ and $\alpha \in \mathbb{R}$ let

$$M(Z,\alpha,\Phi,\mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp(-\alpha m(U) + \Phi(U)), \tag{21}$$

where the infimum is taken over all $\Gamma \subset \bigcup_{k\geq n} \mathcal{U}^k$ such that $\{X(U) : U \in \Gamma\}$ is a cover of Z. When α goes from $-\infty$ to $+\infty$, the quantity in (21) jumps from $+\infty$ to 0 at a unique critical value. Thus, we can define

$$P_Z(\Phi,\mathcal{U}) = \inf\{\alpha \in \mathbb{R} : M(Z,\alpha,\Phi,\mathcal{U}) = 0\}.$$

Using (20), it is shown in [1] that there exists the limit

$$P_Z(\Phi) = \lim_{\operatorname{diam}\mathcal{U}\to 0} P_Z(\Phi,\mathcal{U}).$$

The number $P_Z(\Phi)$ is called the *nonadditive topological pressure* of Φ on the set Z (with respect to f). We emphasize that Z need not be compact nor f-invariant. When Φ is the sequence of function $(\varphi \circ f^n)_{n\in\mathbb{N}}$ for some fixed continuous function $\varphi\colon X \to \mathbb{R}$ the number $P_X(\Phi)$ coincides with the classical topological pressure of the function φ (with respect to f). See [8] for a detailed discussion.

We also need a nonadditive version of the so-called Bowen's equation. Let $\Phi_t = (\varphi_{n,t})_{n\in\mathbb{N}}$ be a sequence of functions $\varphi_{n,t}\colon X \to \mathbb{R}$ for each t in some interval $I \subset \mathbb{R}$. We assume that:

1. Φ_t satisfies (20) for each $t \in I$;
2. there exist constants c_1, $c_2 < 0$, such that if $x \in X$, $n \in \mathbb{N}$, and s, $t \in I$ with $s \neq t$, then

$$c_1 n \leq \frac{\varphi_{n,s}(x) - \varphi_{n,t}(x)}{s-t} \leq c_2 n. \tag{22}$$

Under these assumptions we can establish the following statement, with a slight modification of the proof of Theorem 1.9 in [1].

Proposition 12. *The function $t \mapsto P_Z(\Phi_t)$ is strictly decreasing and Lipschitz, and there exists a unique number $t_P \geq 0$ such that $P_Z(\Phi_{t_P}) = 0$.*

4.2. **Geometric constructions and examples.** Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a continuous transformation and $J \subset \mathbb{R}^m$ an f-invariant set. We assume that J can be decomposed into disjoint sets $R_1, \ldots, R_p$ such that

$$J = \bigcap_{n=1}^{+\infty} \bigcup_{i_1\cdots i_n} R_{i_1\cdots i_n},$$

where for each $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, p\}$ the set $R_{i_1\cdots i_n}$ is an ellipsoid and satisfies

$$R_{i_1\cdots i_n} \cap J = \bigcap_{k=0}^{n-1} f^{-k}(R_{i_{k+1}} \cap J).$$

See Figure 1 for an example with $p = 2$. This is called a *geometric construction* and we refer to J as the *limit set* of the construction (see [1, 8]). One can think of the sets $R_1, \ldots, R_p$ as the elements of a Markov partition of J, although no exponential behavior is assumed here. We also assume that:

1. we can define a function $\chi\colon \{1, \ldots, p\}^{\mathbb{N}} \to J$ by

$$\chi(i_1 i_2 \cdots) = \bigcap_{n=1}^{\infty} R_{i_1\cdots i_n}, \tag{23}$$

   i.e., the intersection in (23) contains one and only one point for each sequence $(i_1 i_2 \cdots) \in \{1, \ldots, p\}^{\mathbb{N}}$;
2. for each $x = \chi(i_1 i_2 \cdots) \in J$ the k-th axis $L_k(x, n)$ of the ellipsoid $R_{i_1\cdots i_n}$ satisfies

$$\Lambda_k^+(x) = \lim_{\delta\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log \sup_{y\in C_x(\delta,n)\cap L_k(x,n)} \frac{d(f^n x, f^n y)}{d(x,y)},$$

   i.e., the axes of each ellipsoid coincide with the directions at which each of the exponents is attained;
3. for each $x = \chi(i_1 i_2 \cdots) \in J$ the images under f of the extreme points of the line segment $R_{i_2\cdots i_n} \cap L_k(fx, n-1)$ are the extreme points of $R_{i_1\cdots i_n} \cap L_k(x, n)$.

We note that since the sets $R_1, \ldots, R_p$ are disjoint, we have $R_{i_1\cdots i_n} \cap R_{j_1\cdots j_n} = \varnothing$ whenever $(i_1 \cdots i_n) \neq (j_1 \cdots j_n)$ and hence χ is injective.

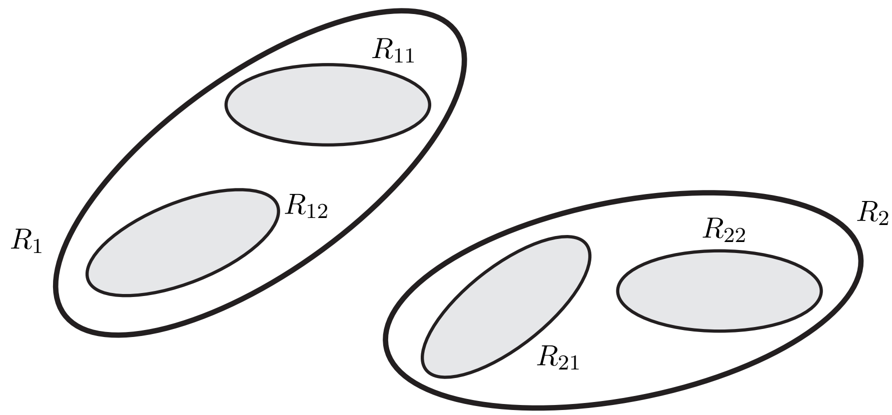

FIGURE 1. The construction of the set J

Under the above assumptions it is easy to verify (by considering the extreme points in condition 3) that

$$\Lambda_k^+(x) \geq \limsup_{\ell\to+\infty} \limsup_{n\to+\infty} \frac{1}{n} \log \frac{a_k(R_{i_{n+1}\cdots i_{n+\ell}})}{a_k(R_{i_1\cdots i_{n+\ell}})} \tag{24}$$

for every $x \in J$. Given a finite f-invariant measure μ on J and $\alpha \geq 0$ we say that J has *distortion at most α with respect to μ* if

$$\Lambda_k^+(x) \leq \limsup_{\ell\to+\infty} \limsup_{n\to+\infty} \frac{1}{n} \log \frac{a_k(R_{i_{n+1}\cdots i_{n+\ell}})}{a_k(R_{i_1\cdots i_{n+\ell}})} + \alpha \tag{25}$$

for μ-almost every $x \in J$.

In particular, one can consider the following simple *differentiable* model as a motivating example. Namely, if f is an expanding map of class $C^{1+\varepsilon}$ and additionally behaves as a product of conformal maps with the directions $L_k(n) = L_k(x,n)$ independent of x for each fixed n, then J has 0-distortion (with respect to any invariant measure). This is a consequence of the property of bounded distortion in each direction $L_k(n)$, in the sense that there exists $M > 0$ such that if $n \in \mathbb{N}$, x, $y \in R_{i_1\cdots i_n}$ and $k = 1, \ldots, m$ then

$$M^{-1} < \frac{\|df_x^n|L_k(n)\|}{\|df_y^n|L_k(n)\|} < M.$$

The above notion of distortion can be though of as a generalization of the property of bounded distortion in the differentiable case, with the parameter α describing the degree of distortion.

We now consider maps that are not necessarily differentiable. In particular we show that there exist:

1. nondifferentiable maps with zero distortion;
2. maps for which (24) is not an identity.

We present a class of examples building on examples constructed in [1].

Example 2. Fix numbers $a > b > 0$ and choose $\delta > 0$ such that $\delta < (a-b)/2$. We now define inductively an increasing sequence $m_n \in \mathbb{N}$. For

each $j \in \mathbb{N}$ let

$$s_j = a \sum_{i \text{ odd} \leq j} m_i + b \sum_{i \text{ even} \leq j} m_i,$$

and $r_j = m_1 + \cdots + m_j$. Fix $m_1 \in \mathbb{N}$. For each $j \in \mathbb{N}$ we choose $m_j \geq m_{j-1}$ such that $|s_j/r_j - a| < \delta/j$ when j is odd, and $|s_j/r_j - b| < \delta/j$ when j is even. Finally, for each $n \in \mathbb{N}$ we define

$$\lambda_n = -s_j - \begin{cases} a(n-n_j) & \text{for } j \text{ odd} \\ b(n-n_j) & \text{for } j \text{ even} \end{cases},$$

where j is the largest positive integer such that $n_j < n$. It is straightforward to verify that

$$\sup\{\lambda_\ell - \lambda_{n+\ell} : \ell > m\} = bn \tag{26}$$

for every n, $m \in \mathbb{N}$. We now consider a geometric construction in the line with $p = 2$ such that each set $R_{i_1\cdots i_n}$ is an interval in $[0,1]$ with length $\exp \lambda_n$ (see Figure 2), and define a map f on J induced by the symbolic dynamics. Namely, for each $x = \chi(i_1 i_2 \cdots) \in J$ we set $fx = \chi(i_2 i_3 \cdots)$.

We also require that each of the sets $R_{i_1\cdots i_n 1}$ and $R_{i_1\cdots i_n 2}$ contain respectively the left and right endpoints of the interval $R_{i_1\cdots i_n}$. This guarantees (see Example 2.6 in [1]) that if x, $y \in J \cap R_{i_1\cdots i_n}$ with $x \neq y$ then

$$\frac{d(f^n x, f^n y)}{d(x,y)} = \frac{e^{\lambda_{m-n}} + \sum_{j=1}^\infty k_j e^{\lambda_{m-n+j}}}{e^{\lambda_m} + \sum_{j=1}^\infty k_j e^{\lambda_{m+j}}},$$

for some $m \in \mathbb{N}$ and $k_j \in \{-1,-1,0,1,2\}$ for each j (although not all sequences k_j are admissible). Using (26) we conclude that

$$\sup_{y \in C_x(\delta,n)} \frac{d(f^n x, f^n y)}{d(x,y)} \geq \sup_m \frac{e^{\lambda_{m-n}}}{e^{\lambda_m}} \times \frac{1 - 2\sum_{j=1}^\infty e^{-bj}}{1 + 2\sum_{j=1}^\infty e^{-bj}} \geq \frac{e^{bn}(1-3e^{-b})}{1+e^{-b}},$$

provided that $b > \log 3$, and the Lyapunov exponent satisfies $\Lambda_1^+(x) \geq b$. On the other hand, using again (26), we have

$$\limsup_{n\to+\infty} \frac{1}{n} \log \frac{a_1(R_{i_{n+1}\cdots i_{n+\ell}})}{a_1(R_{i_1\cdots i_{n+\ell}})} = \limsup_{n\to+\infty} \frac{1}{n} \log \frac{e^{\lambda_\ell}}{e^{\lambda_{n+\ell}}} = b. \tag{27}$$

In view of (24) we conclude that the set J has zero distortion.

It is simple to verify that f cannot be extended to a differentiable map on some open neighborhood of J. Namely, the sequence $\exp(\lambda_n/\lambda_{n+1})$ does not converge but instead oscillates infinitely often between a and b, while when f is differentiable the sequence converges to f' at each point of J.

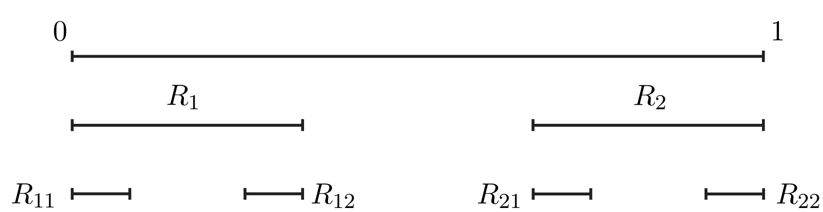

FIGURE 2. The construction of the first example

This example illustrates that even some *nondifferentiable* maps may have zero distortion (in the sense that we can take $\alpha = 0$ in (25)), which in turn allows us to establish sharper dimension estimates.

Example 3. We proceed in a similar way to that in Example 2 to produce a geometric construction with $p = 2$, such that each set $R_{i_1\cdots i_n}$ is an interval in $[0,1]$ with length $\exp\lambda_n$ (with λ_n as in Example 2). Since the identities in (27) are independent of the location of the intervals $R_{i_1\cdots i_n}$ we have

$$\limsup_{\ell\to+\infty}\limsup_{n\to+\infty}\frac{1}{n}\log\frac{a_1(R_{i_{n+1}\cdots i_{n+\ell}})}{a_1(R_{i_1\cdots i_{n+\ell}})} = b.$$

On the other hand, by locating the intervals differently (see Figure 3) we can easily construct examples with $\Lambda_1^+(x) > b$ for some $x \in J$. This shows that (24) may in general be a strict inequality.

Moreover, this example can be effected in such a way that J indeed satisfies (25) for some α. In order to obtain an example it is sufficient to locate the intervals $R_{i_1\cdots i_n}$ as in Example 2 with the exception of those with $i_1 = \cdots = i_n = 1$. Each such $R_{i_1\cdots i_n}$ is located so that its left endpoint does not necessarily coincide with the left endpoint of $R_{i_1\cdots i_{n-1}}$ but instead may be displaced to the right (such as R_{11} in Figure 3). This is done in such a way that at each step n we locate $R_{i_1\cdots i_{n-1}}$, with $i_1 = \cdots = i_{n-1} = 1$, so that

$$b + \frac{\alpha}{2} < \frac{1}{n}\log\frac{d(f^n x, f^n y)}{d(x,y)} < b + \alpha$$

for every x, $y \in R_{i_1\cdots i_n}$ such that $f^n x$ and $f^n y$ are endpoints of some interval $R_{i_1\cdots i_m} \supset R_{i_1\cdots i_n}$ with $m < n$. With this procedure we obtain inductively a geometric construction that satisfies

$$b + \frac{\alpha}{2} < \frac{1}{n}\log\sup_{y\in C_x(\delta,n)}\frac{d(f^n x, f^n y)}{d(x,y)} < b + \alpha$$

for each $n \in \mathbb{N}$ and each distinct x, $y \in R_{i_1\cdots i_n}$. This ensures that J has distortion at most α (and not less than $\alpha/2$).

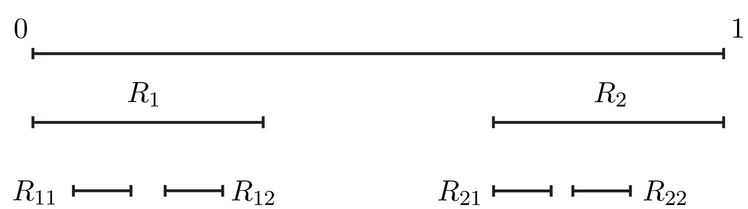

Figure 3. The construction of the second example

The geometric constructions in the real line described in the former examples can be used to obtain "nonconformal" examples in $\mathbb{R}^2$. Consider two geometric construction in the line with intervals $R^1_{i_1\cdots i_n}$ and $R^2_{i_1\cdots i_n}$ respectively. A geometric construction in $\mathbb{R}^2$ can be obtained from these by declaring that the boundary of each set $R_{i_1\cdots i_n}$ is an ellipse with major and minor axis having respectively the lengths of $R^1_{i_1\cdots i_n}$ and $R^2_{i_1\cdots i_n}$.

4.3. **Dimension estimates.** We establish in this section dimension estimates for measures supported on limit sets of geometric constructions. For each $t \in [0, m]$ we define a sequence of functions $\Phi_t = (\varphi_{n,t})_{n\in\mathbb{N}}$ on J by

$$\varphi_{n,t}(x) = nt\alpha - n \sum_{j=1}^{\lfloor t\rfloor -1} \Lambda_j^+(x, n) - n(t - \lfloor t\rfloor)\Lambda_{\lfloor t\rfloor}^+(x, n),$$

where

$$\Lambda_k^+(x, n) = \inf\{\Lambda_k^+(x) : x \in R_{i_1\cdots i_n}\}$$

for each k, and $x = \chi(i_1 i_2 \cdots)$. Since $\varphi_{n,t}$ is constant on each ellipsoid $R_{i_1\cdots i_n}$ the sequence Φ_t satisfies property (20). This family of sequences may however not satisfy in general property (22) and hence we shall use "Bowen's inequalities" instead of Bowen's equations.

Theorem 13. *If the f-invariant set J has distortion at most α with respect to a finite f-invariant measure μ on J then*

$$\dim_H \mu \le \inf\{t \in [0, m] : P_J(\Phi_t) < 0\}.$$

Proof. Let $t \in [0, m]$ be such that $P_J(\Phi_t) < 0$, and fix $\varepsilon \in (0, -P_J(\Phi_t)/2)$. It follows from (25) that for μ-almost every $x = \chi(i_1 i_2 \cdots) \in J$ there exist positive integers $\ell(x)$ and $n(x, \ell)$ such that if $\ell > \ell(x)$ and $n > n(x, \ell)$ then

$$\Lambda_k^+(x) \le \frac{1}{n} \log \frac{a_k(R_{i_{n+1}\cdots i_{n+\ell}})}{a_k(R_{i_1\cdots i_{n+\ell}})} + \alpha + \varepsilon,$$

and hence,

$$\begin{aligned} a_k(R_{i_1\cdots i_{n+\ell}}) &\le \exp[n(-\Lambda_k^+(x) + \alpha + \varepsilon)]a_k(R_{i_{n+1}\cdots i_{n+\ell}}) \\ &\le \exp[n(-\Lambda_k^+(x, n) + \alpha + \varepsilon)] \operatorname{diam} J. \end{aligned} \tag{28}$$

We now consider the set

$$Q_r = \{x \in J : \ell(x) < r \text{ and } n(x, i) < r \text{ for } i \le \ell(x)\}.$$

Clearly

$$\bigcup_{r\in\mathbb{N}} Q_r = J \pmod 0.$$

Fix $r \in \mathbb{N}$. By (28), for each $x = \chi(i_1 i_2 \cdots) \in Q_r$ and $n > r$ we have

$$\begin{aligned} w_t(R_{i_1\cdots i_{n+\ell}}) &\le \exp[\varphi_{n,t}(x) + \varepsilon nt](\operatorname{diam} J)^t \\ &= \exp[\varphi_{n,t}(R_{i_1\cdots i_{n+\ell}}) + \varepsilon nt](\operatorname{diam} J)^t. \end{aligned} \tag{29}$$

For each $\ell \in \mathbb{N}$ we denote by $\mathfrak{U}_\ell$ the cover of J by the ellipsoids $R_{i_1\cdots i_{\ell+1}}$. Take now $\ell > r$ so large that $|P_J(\Phi_t, \mathfrak{U}_\ell) - P_J(\Phi_t, \mathfrak{U})| < \varepsilon$ and thus, by the choice of ε,

$$M(Q_r, -\varepsilon, \Phi_t, \mathfrak{U}_\ell) \le M(J, -\varepsilon, \Phi_t, \mathfrak{U}_\ell) = 0.$$

Hence, for each $\delta > 0$ there exists a cover of Q_r by ellipsoids $R_{I_1}, \ldots, R_{I_N}$ such that: I_j has length $n_j + \ell$ with $n_j > r$ for $j = 1, \ldots, N$; $w_t(R_{I_j})^{1/t} \le \delta$ for $j = 1, \ldots, N$; and

$$\sum_{j=1}^{N} \exp \varphi_{n_j,t}(R_{I_j}) e^{\varepsilon n_j t} < \delta.$$

Using (29) we obtain

$$\sum_{j=1}^{N} w_t(R_{I_j}) \le \sum_{j=1}^{N} \exp \varphi_{n_j,t}(R_{I_j}) e^{\varepsilon n_j t} (\operatorname{diam} J)^t < \delta \max\{1, (\operatorname{diam} J)^m\}.$$

Since δ is arbitrary, we obtain $\mu(Q_r, t) = 0$. It follows from Proposition 11 that $\dim_H Q_r \le t$. Hence,

$$\dim_H \mu \le \dim_H \bigcup_{r\in\mathbb{N}} Q_r = \sup_{r\in\mathbb{N}} \dim_H Q_r \le t$$

whenever $P_J(\Phi_t) < 0$. This establishes the desired result. □

As observed above, under our assumptions the family of sequences Φ_t satisfies property (20) but in general may not satisfy property (22). Assume now that there exist constants c_1, $c_2 < 0$ such that

$$c_1 < \Lambda_k^+(x, n) < c_2$$

for every $x \in J$, $n \in \mathbb{N}$ and $k = 1, \ldots, m$. Then property (22) holds for all sufficiently small α, and by Proposition 12 there exists a unique number $t \ge 0$ such that $P_Z(\Phi_t) = 0$. Furthermore, it follows from Theorem 13 and the monotonicity of $t \mapsto P_J(\Phi_t)$ (see Proposition 12) that $\dim_H \mu \le t$.

We now define

$$\Lambda_k^+ = \operatorname{inf\,ess}\{\Lambda_k^+(x) : x \in J\},$$

where inf ess denotes the essential infimum. The following is now a simple consequence of Theorem 13.

Corollary 14. *If the f-invariant set J has distortion at most α with respect to a finite f-invariant measure μ on J then*

$$\dim_H \mu \le \inf \left\{ t \in [0, m] : \sum_{j=1}^{\lfloor t\rfloor - 1} \Lambda_j^+ + (t - \lfloor t\rfloor)\Lambda_{\lfloor t\rfloor}^+ > \log p + t\alpha \right\}.$$

Proof. For each $t \in [0, m]$ we define a sequence of constant functions $\widetilde{\Phi}_t = (\widetilde{\varphi}_{n,t})_{n\in\mathbb{N}}$ on J by

$$\widetilde{\varphi}_{n,t}(x) = nt\alpha - n \sum_{j=1}^{\lfloor t\rfloor - 1} \Lambda_j^+ - n(t - \lfloor t\rfloor)\Lambda_{\lfloor t\rfloor}^+.$$

Note that this family satisfies property (20). Clearly $\varphi_{n,t}(x) \le \widetilde{\varphi}_{n,t}(x)$ and hence $P_J(\Phi_t) \le P_J(\widetilde{\Phi}_t)$. By Theorem 13 we have $\dim_H \mu \le t$ whenever $P_J(\widetilde{\Phi}_t) < 0$. By Theorem 1.4 and Proposition 1.10 in [1] we have

$$P_J(\widetilde{\Phi}_t) \le \liminf_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp(\widetilde{\varphi}_{n,t}(R_{i_1\cdots i_n})).$$

Therefore

$$P_J(\widetilde{\Phi}_t) \le \liminf_{n\to\infty} \frac{1}{n} \log(p^n \widetilde{\varphi}_{n,t}) = \log p + t\alpha - \sum_{j=1}^{\lfloor t\rfloor - 1} \Lambda_j^+ - (t - \lfloor t\rfloor)\Lambda_{\lfloor t\rfloor}^+,$$

and hence

$$\sum_{j=1}^{\lfloor t\rfloor-1}\Lambda_j^+ + (t-\lfloor t\rfloor)\Lambda_{\lfloor t\rfloor}^+ < \log p + t\alpha.$$

This readily implies the desired statement. □

For example, when $\Lambda_1^+ = \cdots = \Lambda_m^+ = \Lambda^+ > \alpha$, Corollary 14 yields

$$\dim_H \mu \le \frac{\log p}{\Lambda^+ - \alpha}.$$

We can also consider more general symbolic dynamics than the full shift in $\{1,\ldots,p\}^{\mathbb{N}}$. In this case it is straightforward to obtain versions of Theorem 13 and Corollary 14. In fact the statement in Theorem 13 will be unaltered and in the statement of Corollary 14 we must replace $\log p$ by the topological entropy of the dynamics on the set J.

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