Discrete solutions for the porous medium equation with absorption and variable exponents

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Abstract. In this work, we study the convergence of the finite element method when applied to the following parabolic equation:

\[ u_t = \text{div}(|u|^{\gamma(x)} \nabla u) - \lambda |u|^{\sigma(x,t)-2} u + f(x,t), \quad x \in \Omega \subset \mathbb{R}^d, t \in [0,T]. \]

Since the equation may be of degenerate type, we utilise an approximate problem, regularised by introducing a parameter \( \varepsilon \). We prove, under certain conditions on \( \gamma, \sigma \) and \( f \), that the weak solution of the approximate problem converges to the weak solution of the initial problem, when the parameter \( \varepsilon \) tends to zero. The convergence of the discrete solutions for the weak solution of the approximate problem is also proved. Finally, we present some numerical results of a MatLab implementation of the method.
1 Introduction

We shall study the Dirichlet problem for a class of semilinear parabolic equations with variable exponents of nonlinearity. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a domain with Lipschitz-continuous boundary $\partial \Omega$, and let $\Omega_T = \Omega \times [0, T]$ be a cylinder of height $T < \infty$ and $\Gamma_T = \partial \Omega \times [0, T]$. We consider the following problem:

$$
\begin{cases}
  u_t = \text{div}(|u|^{\gamma(x)} \nabla u) - \lambda |u|^\sigma(x,t) - 2u + f(x,t) & \text{in } \Omega_T, \\
  u = 0 & \text{on } \Gamma_T, \\
  u(x,0) = u_0(x) & \text{in } \Omega,
\end{cases}
$$

where $\gamma$ and $\sigma$ are bounded functions defined on $\Omega_T$ such that

$$
\begin{align*}
1 < \gamma^- & \leq \gamma(x) \leq \gamma^+ < \infty, \quad \forall x \in \overline{\Omega}, \\
2 < \sigma^- & \leq \sigma(x,t) \leq \sigma^+ < \infty, \quad \forall (x,t) \in \overline{\Omega_T}.
\end{align*}
$$

If $\gamma$ and $\sigma$ does not depend on $x$ and $t$, then we have the classical porous medium equation with absorption:

$$
\begin{cases}
  u_t = c \Delta(u^m) - \lambda u^p + f(x,t) & \text{in } \Omega_T, \\
  u = 0 & \text{on } \Gamma_T, \\
  u(x,0) = u_0(x) & \text{in } \Omega,
\end{cases}
$$

There is an abundant literature on Problem (4). We refer for example to the papers [9, 10, 8, 7] for one dimensional spatial domains and [16, 17] for higher dimensions. In these papers it is proved that Problem (4), with $m \geq 1$ and $p \geq 0$, has a unique weak solution. The weak solution of (4) has the finite speed of propagation property, and if $0 < p < 1$ then the solution has the finite time extinction property.

Concerning the discrete solutions for the Problem (4) without absorption there are several works with finite elements in space and finite differences in time (see for example [14, 13, 15, 5, 19, 6]).

The convergence of a space-time Galerkin finite element discretization of problem (4) without absorption was proved in [3].

Problems of type (4) were first studied by Antontsev and Shmarev in [11] as a particular case of a more general problem. In 2008, Lian, Gao, Cao and Yuan, [12], studied Problem (1) and proved the existence and uniqueness of weak solutions, but the uniqueness was only proved for $\sigma^- > 2$. In [12], the authors also proved the finite speed of propagation property, the finite time extinction property, for the range $\sigma^+ < 2$, and the localization property, if $\gamma^- + 2 > \sigma^+$. In [14] a moving mesh finite element method for Problem (4) was deduced and
Problem (1) does not, in general, admit classical solutions. A weak solution of Problem (1) is understood as follows.

**Definition 1.** A locally integrable bounded function \( u(x, t) \) is said to be a weak solution of Problem (1) if

(i) \( u \in L_\infty(0, T; L_\infty(\Omega)), \quad |u|^\gamma \nabla u \in L_2(0, T; L_2(\Omega)), \quad u_t \in L_2(0, T; H^{-1}(\Omega)), \)

(ii) \( u = 0 \) on \( \Gamma_T, \)

(iii) for any test-function \( \chi(x, t) \) satisfying the conditions:

\[
\chi \in L_2(0, T; H_0^1(\Omega)) \cap L_\infty(0, T; L_\infty(\Omega)), \quad \chi_t \in L_2(0, T; L_2(\Omega)),
\]

the following integral identity holds:

\[
\int_\Omega ( - u \chi_t + |u|^{\gamma-2} u \chi \nabla \chi + \lambda |u|^{\sigma-2} u \chi - f \chi ) \, d\mathbf{x} \, dt = \int_\Omega u_0 \chi(x, 0) \, d\mathbf{x},
\]

(iv) and \( u(x, 0) = u_0(x) \) in \( \Omega. \)

The existence and uniqueness of weak solutions, in the sense of this definition, was proved in [1]. To the best of the author’s knowledge, there are no results concerning the convergence of the finite element method when applied to problems of this type with variable exponents. Our main result is the derivation of error estimates for numerical approximations to solutions of Problem (1).

The paper is organised as follows: in Section 2, we define an approximate regularized problem and obtain some bounds for its solutions and their derivatives, and we also prove the convergence of the approximate problem to the original problem; in Section 3, we discretize the approximate problem using the Galerkin Method and derive some estimates for the error of the discrete solution; Section 4 is devoted to the numerical analysis of the discrete problem and in Section 5 we present two examples; finally, a summary of the results and outlooks for future research are presented in Section 6.

## 2 Regularisation of the problem

We shall utilise the following approximate problem, regularised by introducing a parameter \( \varepsilon \) in the diffusion and absorption coefficients:

\[
\begin{cases}
    v_t = \text{div}(a_\varepsilon(x, v) \nabla v) - \lambda b_\varepsilon(x, t, v) v + f & \text{in } \Omega_T, \\
    v = 0 & \text{on } \Gamma_T, \\
    v(x, 0) = v_0(x) = u_0(x) & \text{in } \Omega,
\end{cases}
\]
where
\[ a_\varepsilon(x, v) = (v^2 + \varepsilon^2)^{\gamma(x)/2}, \quad b_\varepsilon(x, t, v) = (v^2 + \varepsilon^2)^{\gamma(x, t)/2}, \quad 0 < \varepsilon < 1. \quad (6) \]

It is evident that if \[ \|v\|_{L^\infty} < M \] and \( \gamma \) satisfies (2), then
\[ 0 < \varepsilon^{\gamma^+} \leq a_\varepsilon(x, v) \leq (M^2 + 1)^{\gamma^+} < \infty. \]

We notice that, for \( \sigma > 2 \), the absorption term does not need regularisation, but we will make some simulations with \( \sigma < 2 \). The definition of a weak solution to this problem is similar to the previous one.

According to the classical theory \([11]\), for every \( \varepsilon > 0 \), the regularised problem has a weak solution \( v \) which satisfies this definition. Moreover, if \( \gamma \in H^2(\Omega) \), then \( v_t, \Delta v \in L^2(\Omega_T) \) and the weak solution also satisfy (5). If \( \gamma \in C^{2+\alpha}(\bar{\Omega}) \), then the solution is classical, in the sense that \( v_t, D^2v \in C^\alpha(\Omega_T) \). The proofs in this work are adaptations of the proofs done in \([3]\) and almost all the arguments are valid for this problem. We will only present the main modifications. In what follows, we will only refer to the conditions on the regularity of \( \gamma \) needed to prove each estimate. Since the influence of the parameter \( \varepsilon \) on the regularity of the solutions of Problem (5) is not clear, we begin with a collection of basic regularity results. In what follows, \( C \) will represent a constant, but not always the same value.

**Theorem 2.** Let \( \gamma \) be a measurable function in \( \Omega \) which satisfies condition (2). If
\[ \|u_0\|_{L^\infty(\Omega)} + \int_0^T \|f\|_{L^\infty(\Omega)} \, dt < C, \quad (7) \]
then the solution \( v \) of Problem (5) satisfies
\[ \|v(x, t)\|_{L^\infty(\Omega)} \leq \|v(x, 0)\|_{L^\infty(\Omega)} + \int_0^T \|f\|_{L^\infty(\Omega)} \, dt \leq C, \quad t \in [0, T], \quad (8) \]
where \( C \) does not depend on \( \varepsilon \).

**Proof.** Multiplying the first equation of (5) by \( v^{2k-1} \) and integrating over \( \Omega \) the absorption term become a nonnegative term on the left hand side and so it can be ignored. The rest is identical of the proof of Theorem 2.2 in \([3]\). \( \square \)

**Theorem 3.** Suppose that \( v \) is a solution of (5) with \( \gamma \in H^1(\Omega) \) and that the conditions of the previous theorem are satisfied. Then
\[ \int_0^T \|\nabla v\|_{L^2(\Omega)}^2 \, dt \leq C\varepsilon^{-\gamma^+}, \]
where \( C \) does not depend on \( \varepsilon \).
Proof. See Theorem 2.3 in [3].

**Theorem 4.** Suppose that $\gamma \in H^2(\Omega)$ and $\nabla u_0 \in L_2(\Omega)$. If $v$ is a solution of (5) and the conditions of Theorem 2 are satisfied, then
\[
\int_0^T \|\Delta v\|_{L_2(\Omega)}^2 \, dt \leq C\varepsilon^{-8\gamma^+},
\]
where $C$ does not depend on $\varepsilon$.

Proof. We refer to Theorem 2.4 in [3]. Notice that the additional term in $g$ is easily handled using Theorem 2.

**Theorem 5.** Suppose that $\gamma \in H^2(\Omega)$, $v$ is a solution of (5), condition (7) is satisfied and $\nabla u_0 \in L_2(\Omega)$. Then
\[
\int_0^T \|v_t\|_{L_2(\Omega)}^2 \, dt \leq C\varepsilon^{-2\gamma^+},
\]
where $C$ does not depend on $\varepsilon$.

Proof. See Theorem 2.5 in [3], and the additional term can also be easily handled using Theorem 2.

Our next task is to show that $v$ is close to $u$ in an appropriate norm.

**Lemma 6.** If $\gamma$ satisfies (2), $\sigma$ satisfies (3) and $v$ is bounded, then, for every $0 < \varepsilon < 1$, we have
\[
||v|^{\gamma(x)}_\gamma(v^2 + \varepsilon^2)^{-\frac{\gamma(x)}{2}}| \leq C\varepsilon.
\]

(9)

\[
||v|^{\sigma(x,t)-2}_\sigma(v^2 + \varepsilon^2)^{-\frac{\sigma(x,t)-2}{2}}| \leq C\varepsilon.
\]

(10)

Proof. For $\gamma$, the result is proved in Lemma 2.6 in [3]. Concerning (10) we have that
\[
||v|^{\sigma(x,t)-2}_\sigma(v^2 + \varepsilon^2)^{-\frac{\sigma(x,t)-2}{2}}| = \left| \int_\varepsilon^0 (\sigma - 2)\tau v(v^2 + \tau^2)^{-\frac{\sigma-4}{2}} \, d\tau \right|
\]
\[
\leq \int_0^\varepsilon (\sigma - 2)(v^2 + \tau^2)^{-\frac{\sigma-2}{2}} \, d\tau
\]
\[
\leq C\varepsilon
\]

\]

\]

\]
Lemma 7. Let $u$ be a weak solution of (7) and $v$ a weak solution of (5). The function $w = u - v$ satisfies the equation

$$\int_{\Omega_T} w(-\chi_t - A\Delta \chi + B\nabla \chi + Q\chi) \, dx \, dt =$$

$$\int_{\Omega_T} (F[v] - F_\varepsilon[v])\Delta \chi + (G[v] - G_\varepsilon[v])\nabla \chi + (J[v] - J_\varepsilon[v])\chi \, dx \, dt \quad (11)$$

with

$$A = \frac{1}{\gamma + 1} \frac{u|u|^\gamma - v|v|^\gamma}{u - v}, \quad B = A\nabla \ln(\gamma + 1) - D \quad (12)$$

and

$$Q = \lambda \frac{u|u|^{\sigma - 2} - v|v|^{\sigma - 2}}{u - v} \quad (13)$$

where

$$D = \frac{\nabla \gamma}{2(\gamma + 1)} \left( \frac{u|u|\gamma\ln(u^2) - v|v|\gamma\ln(v^2)}{u - v} \right). \quad (14)$$

Proof. Defining

$$J(x, t) = J[u] = \lambda|u|^{\sigma - 2}u \quad \text{and} \quad J_\varepsilon(x, t) = J_\varepsilon[u] = \lambda(u^2 + \varepsilon^2)^{\frac{\sigma - 2}{2}}u.$$ 

Writing


and $J[u] - J[v] = Qw$, with $Q$ defined in (13), we obtain (11). \qed

Lemma 8. Let $\eta(x, t)$ be the solution of the following parabolic problem:

$$\begin{cases}
\eta_t - (A + \varepsilon)\Delta \eta + B\nabla \eta + Q\eta = \phi & \text{in} \quad \Omega_T, \\
\eta(x, 0) = 0 & \text{in} \quad \Omega, \\
\eta(x, t) = 0 & \text{on} \quad \partial \Omega,
\end{cases} \quad (15)$$

where $\varepsilon > 0$ is an arbitrary small parameter, $\phi \in L^2(\Omega_T)$ is an arbitrary function, and $A, B, Q$ are defined in (12), (13) and (14). If $\sigma - 2 > \frac{1}{2}$ and

$$\frac{\phi^2}{A + \varepsilon} \leq C, \quad (16)$$

then

$$\int_\Omega |\nabla \eta|^2 \, dx + \int_0^t \int_\Omega (A + \varepsilon)(\Delta \eta)^2 \, dx \, dt \leq C. \quad (17)$$
Proof. Using the arguments of the proof in Lemma 2.8 of [3] we only have to obtain a bound for the term $\frac{Q^2}{A+\epsilon}$. Writing

$$\frac{Q^2}{A+\epsilon} \leq \frac{Q^2}{A} = \left(\int_0^1 |\xi u + (1 - \xi)|^{\sigma-2} d\xi\right)^2 \leq \int_0^1 |\xi u + (1 - \xi)|^{2\sigma-4-\gamma} d\xi \leq C$$

with $C$ depending only on $\gamma^-, \sigma^+, \sup u$ and $\sup v$, we obtain the required result.

Theorem 9. Let $u$ be a weak solution of (1) and $v$ a weak solution of (5). If

$$1 < \gamma^- \leq \gamma(x) \leq \gamma^+ \text{ in } \bar{\Omega} \text{ and } \sup_{x \in \Gamma} |\nabla \gamma| < \infty,$$

then

$$\|u - v\|_{L_{\gamma^+}^{\gamma^++1}(\Omega_T)} \leq C\varepsilon^{\frac{1}{2}}, \tag{18}$$

where $C$ does not depend on $\varepsilon$.

Proof. In the proof of Lemma 2.8 we can use Lemma [10] to prove that

$$\int_{\Omega_T} (|J[v] - J_\varepsilon[v]|) \eta \, dx \, dt \leq$$

$$\leq \sup(|J[v] - J_\varepsilon[v]|) \sqrt{T \text{meas}(\Omega)} \left(\int_{\Omega_T} \eta^2 \, dx \, dt\right)^{\frac{1}{2}} \leq$$

$$\leq C\varepsilon \left(\int_{\Omega_T} |\nabla \eta|^2 \, dx \, dt\right)^{\frac{1}{2}} \leq$$

$$\leq C\varepsilon.$$
3 Discrete solution

In this section, we follow [18] and apply the continuous Galerkin method in the space variable and the discontinuous Galerkin method in the time variable to the approximate problem. For simplicity, we will restrict our study to $\mathbb{R}^2$, but the results can be easily generalized to any space dimension.

Suppose that $h$ is a positive constant and let $T_h$ denote a regular partition of $\Omega$ into disjoint triangles $T_k$, see [2] for more details. Now let $S_{hr}$ denote the set of continuous functions on the closure of $\Omega$, which are polynomials of degree $r$, in each triangle of $T_h$ and which vanish on $\partial \Omega$, that is,

$$S_{hr} = \{ w(\mathbf{x}) \in C_0^0(\bar{\Omega}) | w|_{T_k} \text{ is a polynomial of degree } r \text{ for all } T_k \in T_h \}.$$ 

In the same way, consider $\delta > 0$, the partition $[0, T] = \cup_{n=0}^{N-1} I_n, I_n = [t_n, t_{n+1}], t_{n+1} = t_n + \delta$ and the space

$$S_{hr}^{\delta s} = \{ W = W_{h\delta} : [0, +\infty[ \rightarrow S_{hr} | W|_{I_n} = \sum_{n=0}^{s} t^n w_n(\mathbf{x}), w_n \in S_{hr} \}.$$ 

We do not impose continuity at the nodal points, $t_n$, for those functions but they must be continuous to the left of these. For $W \in S_{hr}^{\delta s}$, we denote by $W_n$ and $W_n^+$ the value of $W$ and its limit from above at $t_n$, respectively. By $[W]_n$ we mean the jump of $W$ in $t_n$, defined as $[W]_n = W_n^+ - W_n$. We seek a discrete approximation $v \approx V \in S_{hr}^{\delta s}$ that satisfies the definition of a weak solution.

**Definition 10.** A function $V \in S_{hr}^{\delta s}$ is said to be a discrete solution of the approximate problem if $V = 0$ on $\partial \Omega$ and satisfies the following equation, for all $W \in S_{hr}^{\delta s}$,

$$\sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} V_t W \, d\mathbf{x} \, dt + \sum_{n=1}^{N-1} \int_{\Omega} [V]_n W_n \, d\mathbf{x} + \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} a_{\epsilon}(\mathbf{x}, V) \nabla V \cdot \nabla W \, d\mathbf{x} \, dt +$$

$$+ \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} b_{\epsilon}(\mathbf{x}, t, V) V W \, d\mathbf{x} \, dt = \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} f W \, d\mathbf{x} \, dt,$$  

where $V_t$ is the piecewise polynomial of degree $s - 1$ which interpolates $\frac{\partial V}{\partial t}$ in each $I_n$.

Since $W$ is not required to be continuous at $t_n$, we may choose $W$ to vanish outside $I_n$. Hence, Equation (19) reduces to $N$ equations, one for each $I_n$. 


Then the discrete problem consists in finding \( V \in S^{δs}_{hr} \) such that for all \( W \in S^{δs}_{hr} \), and for each \( n \in \{0, \ldots, N\} \),

\[
\int_{I_n} \int_{Ω} V W \, dx \, dt + \int_{Ω} V_{n-1}^{+} W_{n-1}^{+} \, dx + \int_{I_n} \int_{Ω} a_ε(x, V) \nabla V \cdot \nabla W \, dx \, dt + \\
+ \int_{I_n} \int_{Ω} b_ε(x, t, V) W \, dx \, dt = \int_{I_n} \int_{Ω} f W \, dx \, dt + \int_{Ω} V_{n-1}^{+} W_{n-1}^{+} \, dx.
\] (20)

**Theorem 11.** If \( V_{n-1} \in L_2(Ω) \) and \( f \in L_2(Ω \times I_n) \), then Problem (20) has a solution \( V \).

**Proof.** Let \( n \geq 1 \) be fixed. For each \( h, δ > 0 \), we define the continuous mapping \( F : S^{δs}_{hr} \rightarrow S^{δs}_{hr} \) by

\[
\int_{I_n} \int_{Ω} F(V) W \, dx \, dt = \int_{I_n} \int_{Ω} V W \, dx \, dt + \int_{Ω} V_{n-1}^{+} W_{n-1}^{+} \, dx + \\
+ \int_{I_n} \int_{Ω} b_ε(x, t, V) W \, dx \, dt + \int_{I_n} \int_{Ω} a_ε(x, V) \nabla V \cdot \nabla W \, dx \, dt - \\
- \int_{I_n} \int_{Ω} f W \, dx \, dt - \int_{Ω} V_{n-1}^{+} W_{n-1}^{+} \, dx, \quad \forall W \in S^{δs}_{hr}.
\]

Choosing \( W = V \), using the lower bound of \( a_ε \) and \( b_ε \) and applying the Holder inequality, we have

\[
\int_{I_n} \int_{Ω} F(V) V \, dx \, dt \geq \frac{1}{2} \int_{I_n} \frac{d}{dt} \| V \|^2_{L_2(Ω)} \, dt + \| V_{n-1}^{+} \|^2_{L_2(Ω)} + \\
\varepsilon^{\sigma+2} \int_{I_n} \| V \|^2_{L_2(Ω)} \, dt + \varepsilon^{\gamma} \int_{I_n} \| \nabla V \|^2_{L_2(Ω)} \, dt - \\
- \int_{I_n} \| f \|_{L_2(Ω)} \| V \|_{L_2(Ω)} \, dt - \| V_{n-1} \|_{L_2(Ω)} \| V_{n-1}^{+} \|_{L_2(Ω)}.
\]

Hence, integrating the first term in \( t \), using the Cauchy inequality and the Poincare inequality, we arrive at

\[
\int_{I_n} \int_{Ω} F(V) V \, dx \, dt \geq \frac{1}{2} \| V_n \|^2_{L_2(Ω)} + C_ε^{\max{\{\gamma^+, \sigma^+\}}} \int_{I_n} \| V \|^2_{L_2(Ω)} \, dt - \\
- \int_{I_n} \| f \|_{L_2(Ω)} \| V \|_{L_2(Ω)} \, dt - \frac{1}{2} \| V_{n-1} \|^2_{L_2(Ω)}.
\]
Applying once again the Cauchy inequality, we conclude that
\[
\int_{I_n} \int_{\Omega} F(V) V \, dx \, dt \geq \frac{1}{2} ||V_n||_{L^2(\Omega)}^2 + \frac{C_{\epsilon} \max \{\gamma^+, \sigma^+ - 2\}}{2} \int_{I_n} ||V||_{L^2(\Omega)}^2 \, dt - \frac{1}{2C_{\epsilon} \max \{\gamma^+, \sigma^+ - 2\}} \int_{I_n} ||f||_{L^2(\Omega)}^2 \, dt - \frac{1}{2} ||V_{n-1}||_{L^2(\Omega)}^2.
\]
If \( V \) belongs to \( \Lambda = \{ W \in S_{\delta s}^{r} : ||W||_{L^2(\Omega \times I_n)} \leq \epsilon \} \),
\[
\epsilon > \frac{1}{C_{\epsilon} \max \{\gamma^+, \sigma^+ - 2\}} ||V_{n-1}||_{L^2(\Omega)} + \frac{1}{(C_{\epsilon} \max \{\gamma^+, \sigma^+ - 2\})^2} ||f_n||_{L^2(\Omega \times I_n)},
\]
then \( \int_{\Omega} F(V) V \, dx \geq 0, \forall V \in \partial \Lambda \). A corollary to Brouwer’s Fixed-Point Theorem implies the existence of \( V_\epsilon \in B \) such that \( F(V_\epsilon) = 0 \), and the result is proved with \( V = V_\epsilon \).

We terminate this section with an error estimate for the fully discrete scheme (19). The proof is very similar to that of Theorem 9.

**Theorem 13.** Let \( v \) be a solution of Problem (5) and \( V \) a solution of problem (19). If \( \gamma \in H^2(\Omega) \) and the conditions of Theorem 2 are satisfied, then
\[
||v - V||_{L^{\gamma+1}_{r+1}(\Omega \times I_n)} \leq C_{\epsilon} \gamma^+(h^{r+1}||v||_{L^\infty(0,T;H^{r+1}(\Omega))} + \delta^{s+1}||v||_{W^{s+1,\infty}(0,T;L^2(\Omega))}),
\]
where \( C \) does not depend on \( \epsilon, h, r, \delta \) and \( s \).
From the previous theorem, Theorem 9 and the estimates for the solution $v$ and its derivatives, we can estimate the error between $u$ and $V$ using the triangle inequality:

$$\|V - u\|_{L^{r+1}(\Omega_T)} \leq C\varepsilon^{-a r^+} h^{r+1} + C\varepsilon^{-b r^+} \delta^{s+1} + C\varepsilon^{\frac{1}{2}}.$$

For suitable choices of $r$, $s$, $h$ and $\delta$, we can prove that the error vanishes when $\varepsilon$ tends to zero.

### 4 Numerical implementation

Finally, we are concerned with the numerical solution of the discretised problem. Obviously, there is no need to solve the problem simultaneously on all time intervals, since the problem on $I_n$ is only dependent on the information from $I_{n-1}$, but, for each $I_n$, we need to solve a nonlinear system of equations.

Because of limitations concerning the regularity of the solution, we will only consider polynomials of degree zero in time. This simplifies the equations since $V_t = 0$, $V_n - 1 = V_n$ and $W_n - 1 = W_n$. Denoting $f_n(x) = \frac{1}{\delta} \int_{I_n} f \, dt$, we have to solve the equations

$$\int_{\Omega} V_n W_n \, dx + \delta \int_{\Omega} a_\varepsilon(x, V_n) \nabla V_n \cdot \nabla W_n \, dx + \delta \int_{\Omega} b_\varepsilon(x, t_n, V_n) V_n W_n \, dx = \int_{\Omega} f_n W_n \, dx + \int_{\Omega} V_{n-1} W_n \, dx$$

for all $W_n \in S^{60}_h$, $n \in \{1, \ldots, N\}$. In (22), we have a nonlinear system of equations which must be solved in each time interval. We could use several methods, such as a Newton type method or some linearisation but we choose the Fixed Point Method, because it is efficient and easy to analyse and implement. To find $V_n$, we use the following algorithm.

Given $V_{n-1} \in S^{60}_h$, $f_n \in L_2(\Omega)$ and $tol > 0$, we define $k = 0$ and $W_0 = V_{n-1}$.

1) For $k \geq 1$, $W_k \in S^{60}_h$ is calculated such that for all $W \in S^{60}_h$,

$$\int_{\Omega} W_k W \, dx + \delta \int_{\Omega} a_\varepsilon(x, W_{k-1}) \nabla W_k \cdot \nabla W \, dx +$$

$$+ \delta \int_{\Omega} b_\varepsilon(x, t_n, W_{k-1}) W_k W \, dx = \int_{\Omega} f_n W \, dx + \int_{\Omega} W_0 W \, dx. \quad (23)$$
2) If $\|W_k - W_{k-1}\| \geq tol$, set $k = k + 1$ and go to 1).

3) If $\|W_k - W_{k-1}\| < tol$, set $V_n = W_k$ and terminate.

Before proving the convergence of the algorithm, we have to prove the stability by showing that all the estimates are bounded.

**Lemma 14.** Assuming that $\|V_{n-1}\|_{L^2(\Omega)} \leq C$, where $C$ does not depend on $n$, and $f_n \in L^2(\Omega)$, there exists $0 < C' < \infty$ such that

$$\|W_k\|_{L^2(\Omega)} \leq C', \quad \forall k > 0,$$

where $C'$ does not depend on $n$ or $k$.

**Proof.** If $W = W_k$ in (23), then

$$\int_{\Omega} W_k^2 \, dx + \delta \int_{\Omega} a_{\varepsilon}(x, W_{k-1})(\nabla W_k)^2 \, dx + \delta \int_{\Omega} b_{\varepsilon}(x, t_n, W_{k-1}) W_k^2 \, dx =$$

$$= \int_{\Omega} f_n \, W_k \, dx + \int_{\Omega} W_0 \, W_k \, dx.$$

Since the second and third term are nonnegative, using Holder’s inequality, we have

$$\|W_k\|^2_{L^2(\Omega)} \leq \|f_n\|_{L^2} \|W_k\|_{L^2} + \|W_0\|_{L^2} \|W_k\|_{L^2}.$$

Simplifying we obtain $\|W_k\|_{L^2(\Omega)} \leq \|f_n\|_{L^2} + \|W_0\|_{L^2}$. □

The convergence is proved using the $L^2$ contraction and limiting the length of the time intervals.

**Theorem 15.** Let $tol > 0$. There exists $C > 0$ such that if $\delta < \frac{1}{C(h^{-2}\varepsilon^{-\gamma^+} + \varepsilon^{\gamma^+})}$, then there exists $k^* \in N$ so that

$$\|W_k - W_{k-1}\|_{L^2(\Omega)} < tol, \quad \forall k > k^*.$$

**Proof.** In order to adapt the proof of Theorem 4.2 in [3], we just need to write

$$b_{\varepsilon}(x, t_n, W_{k-1}) W_k - b_{\varepsilon}(x, t_n, W_{k-2}) W_{k-1} = b_{\varepsilon}(x, t_n, W_{k-1}) E_k +$$

$$b_{\varepsilon}(x, t_n, W_{k-1}) W_{k-1} - b_{\varepsilon}(x, t_n, W_{k-2}) W_{k-2} - b_{\varepsilon}(x, t_n, W_{k-2}) E_{k-1} =$$

$$b_{\varepsilon}(x, t_n, W_{k-1}) E_k + Q E_{k-1} - b_{\varepsilon}(x, t_n, W_{k-2}) E_{k-1}$$

and the result is established with a similar $q = (C\delta(h^{-2}\varepsilon^{-\gamma^+} + \varepsilon^{\gamma^+}))^{\frac{1}{2}} < 1$. □
5 Examples

5.1 Example 1

We simulated problem [1] with
\[ \gamma = x^2 + y^2 + 1, \quad \sigma = 2 + \frac{x^2 + y^2}{t + 1} \quad \text{and} \quad u_0 = 0.25 - x^2 - y^2. \]

The function \( f \) is defined such that
\[ u = \frac{1 - 4x^2 - 4y^2 + t^2}{4t^2 + 4} \]
is the exact explicit solution of the problem. We used approximations of degree 1 in space and the mesh used was a structured mesh with 1600 triangles, as shown in Figure [1]

![Spatial mesh](image)

Figure 1: Spatial mesh

In Figure [2] we present the solution obtained with the parameters \( h = 10^{-1} \), \( \delta = 10^{-2} \) and \( \epsilon = 10^{-8} \).

Knowing an explicit solution permitted us to calculate the exact error and to compare the numerical results with the theoretical results. We simulated the problem with several combinations of the parameters \( \epsilon, h \) and \( \delta \) and we calculated the \( L_2 \) norm error of the solution at \( t = 1 \). The results were collected in Figures [3] [4] and [5]. We conclude that, in this case, the numerical order of convergence for \( h \) is approximately 1.5, for \( \delta \) approximately 1, and for \( \epsilon \) approximately 1.
Figure 2: Evolution of the solution in some values of $t$

Figure 3: Study of the convergence for $h$.

Figure 4: Study of the convergence for $\delta$.

Figure 5: Study of the convergence for $\varepsilon$.

5.2 Example 2

Finally, we simulated an example with

$$\gamma = 3 + \frac{3x - 3y}{t + 2} \quad \text{and} \quad \sigma = 2 + \frac{x + y}{t + 2},$$

and with the initial data $u_0$ defined by

$$u_0 = \begin{cases} 
1 - 4x^2 - 4y^2, & \sqrt{x^2 + y^2} < 0.5 \\
0, & \text{rest of } \mathbb{R}^2
\end{cases}.$$  

We simulated with approximations of degree 2. We chose the domain to be $[-1, 1] \times [-1, 1]$. The parameters of the method were $h = 10^{-1}$, $\delta = 10^{-2}$.
and $\varepsilon = 10^{-8}$, and the mesh used was the same as in the previous example. We stopped the simulation at $t = 1$. In Figure 6 we represent the evolution of the solution.

![Figure 6: Evolution of the solution in some values of $t$](image)

In the figures, we observe a region where the waiting time effect is present and a region where the boundary starts to move at $t = 0$. We can also observe a region with fast diffusion and a region with slow diffusion, as expected.

6 Conclusions

In this work we proved, under certain conditions on the exponents, the convergence of a space-time Galerkin finite element method for a porous media equation with absorption and variable exponents. The numerical computations agree with the theoretical results. The case where $\gamma$ depends on $t$ and the case where $\sigma < 2$ are under study.
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REFERENCES


