Singular Dressing Actions on Harmonic Maps

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Abstract

In this paper we prove that any harmonic map ϕ from a two-sphere S^2 into an arbitrary compact semisimple matrix Lie group G may be reduced to a constant by using the *singular dressing actions* introduced in [1]; this reduction induces a factorization of ϕ into flag factors $\mathrm{S}^2 \to G$, and the singular dressing actions are produced from curves of *simple factors* (rational loops having a minimum number of singularities, whose dressing action can be computed explicitly) for $G^{\mathbb{C}}$. A version of this result for an arbitrary inner symmetric space G/K is established. We also prove generating theorems for the rational loops of the fundamental representations of $\mathrm{Sp}(n)^{\mathbb{C}}$ and $\mathrm{SU}(n)^{\mathbb{C}}$: in both cases the class of generators is slightly larger than the class of simple factors.

Keywords: Harmonic maps, dressing actions, simple factors, loop groups, symmetric spaces.

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1 Introduction

In the seminal paper [18], Uhlenbeck observed that, in the setting of harmonic maps of $\mathbb{C}$ into a compact Riemannian symmetric space G/K, the harmonic map equations amount to the flatness of a family of connections depending on an auxiliary parameter $\lambda \in \mathrm{S}^1$. This zero-curvature formulation yields an action of a certain loop group on the space of harmonic maps; underlying this *dressing action* is the existence of Iwasawa-type decompositions of the loop groups and loop algebras concerned. At same time, Uhlenbeck introduced the fundamental procedure of *adding a uniton*, which is another way of obtaining new harmonic maps from known ones, and proved that all harmonic two-spheres in the unitary group can be factorized into finite products of *flag factors* $\mathrm{S}^2 \to \mathrm{U}(n)$ by adding unitons. This was subsequently generalized in [4] to the case of an arbitray compact semisimple Lie group G.

Bergvelt and Guest [1], again inspired in [18], exploited the singularities of the dressing action in order to enlarge the corresponding orbits of harmonic maps – this limiting process is called the *modified completion procedure* or the *singular dressing action* [1, 12]. They proved that any harmonic map from the two-sphere S^2 into the complex projective space $\mathbb{C}\mathrm{P}^{\mathrm{n}}$ may be reduced to a constant by applying twice this procedure. Subsequently, Jiao [12] proved that any harmonic map ϕ from S^2 into the unitary group $\mathrm{U}(n)$ may be reduced to a constant by applying n singular dressing actions. This reduction induces a factorization of ϕ into flag factors $\mathrm{S}^2 \to \mathrm{U(n)}$, and the n singular dressing actions are produced from curves $\{\gamma_a\}$ of rational loops of the form $\gamma_a(\lambda) = \pi_V^\perp + \zeta_a(\lambda)\pi_V$, where

$$\zeta_a(\lambda) = \frac{\lambda - a}{\overline{a}\lambda - 1}\,\frac{\overline{a} - 1}{1 - a}. \tag{1}$$

These rational loops are precisely the *simple factors* of Uhlenbeck and they generate the group of rational loops in the matrix Lie group $\mathrm{Gl}(n, \mathbb{C})$ satisfying the reality condition with respect to $\mathrm{U}(n)$ [18]. In [7], the authors gave a consistent definition of simple factor for an arbitrary complex reductive Lie group and an arbitrary representation, and proved that, in the $\mathrm{SO}(n)^{\mathbb{C}}$ and $\mathrm{G}_2^{\mathbb{C}}$ cases, with respect to the fundamental representations, the simple factors generate the group of rational loops satisfying the reality condition.

In this paper we prove (Theorem 4) that any harmonic map ϕ from S^2 into an arbitrary compact semisimple matrix Lie group G may be reduced to a constant by applying a finite number of singular dressing actions; this reduction induces a factorization of ϕ into flag factors $\mathrm{S}^2 \to \mathrm{G}$, and the singular dressing actions are produced from curves of simple factors for $G^{\mathbb{C}}$. A version of this result for an arbitrary inner symmetric space G/K is established (Theorem 6). We also prove generating theorems for the rational loops of the fundamental representations of $\mathrm{Sp}(n)^{\mathbb{C}}$ (Theorem 2) and $\mathrm{SU}(n)^{\mathbb{C}}$ (Theorem 3): in both cases the class of generators is slightly larger than the class of simple factors.

2 Preliminaries

2.1 Complex Extended Solutions

We start by reviewing the well known [18] reformulation of harmonicity equations, for maps into a compact symmetric space, in terms of loops of flat connections.

Let G be a compact (connected) semisimple matrix Lie group, with identity e and Lie algebra $\mathfrak{g}$. Equip G with a bi-invariant metric. Let $G^{\mathbb{C}}$ be the complexification of G, with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. Consider the based loop group $\Omega G = \{\gamma : \mathrm{S}^1 \to G \text{ (smooth)} \mid \gamma(1) = e\}$ and the corresponding infinite-dimensional Lie algebra $\Omega\mathfrak{g} = \{\gamma : \mathrm{S}^1 \to \mathfrak{g} \text{ (smooth)} \mid \gamma(1) = 0\}$. A smooth map $\Phi : \mathbb{C} \to \Omega G$ is called an *extended solution* if it satisfies $\Phi^{-1}d\Phi = (1-\lambda^{-1})\alpha' + (1-\lambda)\alpha''$ for each $\lambda \in \mathrm{S}^1$, where α' is a $\mathfrak{g}^{\mathbb{C}}$-valued $(1,0)$-form on $\mathbb{C}$ with complex conjugate α''. Observe that, for each $z \in \mathbb{C}$, $\lambda \mapsto \Phi(z)(\lambda)$ is holomorphic on $\mathbb{C}^*$. If $\Phi : \mathbb{C} \to \Omega G$ is an extended solution, $\phi : \mathbb{C} \to G$ defined by $\phi(z) = \Phi(z)(-1)$ is harmonic; conversely, if $\phi : \mathbb{C} \to G$ is harmonic, there exists an extended solution $\Phi : \mathbb{C} \to \Omega G$, unique up to multiplication on the left by an element $\gamma \in \Omega G$ satisfying $\gamma(-1) = e$, such that $\phi(z) = \Phi(z)(-1)$, for all $z \in \mathbb{C}$.

2.2 Holomorphic Potentials

In [8], the authors introduced a general recipe to produce all harmonic maps from $\mathbb{C}$ to G out of certain holomorphic potentials. Next we reformulate this procedure according to our needs. See also [6] for details.

Fix $0 < \varepsilon < 1$. Let C_ε and $C_{1/\varepsilon}$ denote the circles of radius ε and $1/\varepsilon$ centered at $0 \in \mathbb{C}$; define open subsets of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ by $I_\varepsilon = \{\lambda \in \mathbb{P}^1 \mid |\lambda| < \varepsilon\}$, $I_{1/\varepsilon} = \{\lambda \in \mathbb{P}^1 \mid |\lambda| > 1/\varepsilon\}$, and $E^\varepsilon = \{\lambda \in \mathbb{P}^1 \mid \varepsilon < |\lambda| < 1/\varepsilon\}$; put $I^\varepsilon = I_\varepsilon \cup I_{1/\varepsilon}$ and $C^\varepsilon = C_\varepsilon \cup C_{1/\varepsilon}$ so that $\mathbb{P}^1 = I^\varepsilon \cup C^\varepsilon \cup E^\varepsilon$. Consider the infinite-dimensional Lie groups

$$\begin{aligned}
\Lambda^\varepsilon G &= \{\gamma : C^\varepsilon \to G^{\mathbb{C}} \text{ (smooth)} \mid \overline{\gamma(\lambda)} = \gamma(1/\bar{\lambda})\} \\
\Omega^\varepsilon_E G &= \{\gamma \in \Lambda^\varepsilon G \mid \gamma \text{ extends holomorphically to } \gamma : E^\varepsilon \to G^{\mathbb{C}} \text{ and } \gamma(1) = e\} \\
\Lambda^\varepsilon_I G &= \{\gamma \in \Lambda^\varepsilon G \mid \gamma \text{ extends holomorphically to } \gamma : I^\varepsilon \to G^{\mathbb{C}}\}
\end{aligned}$$

and the corresponding infinite-dimensional Lie algebras $\Lambda^\varepsilon\mathfrak{g}$, $\Omega^\varepsilon_E\mathfrak{g}$, and $\Lambda^\varepsilon_I\mathfrak{g}$.

The loop group $\Lambda^\varepsilon G$ admits a Iwasawa-type decomposition [13]: multiplication

$$\Omega^\varepsilon_E G \times \Lambda^\varepsilon_I G \to \Lambda^\varepsilon G \tag{2}$$

is a diffeomorphism. In particular, any $\gamma \in \Lambda^\varepsilon G$ may by written uniquely in the form $\gamma = \gamma_E \gamma_I$, where $\gamma_E \in \Omega^\varepsilon_E G$ and $\gamma_I \in \Lambda^\varepsilon_I G$.

Remark 1. Define $\Lambda G^{\mathbb{C}} = \{\gamma : S^1 \to G^{\mathbb{C}} \mid \gamma \text{ is smooth}\}$ and

$$\Lambda_+ G^{\mathbb{C}} = \{\gamma \in \Lambda G^{\mathbb{C}} \mid \gamma \text{ extends holomorphically to } |\lambda| < 1\ \}.$$

The limiting case of (2) as $\varepsilon \to 1$ is the familiar decomposition [15]

$$\Omega G \times \Lambda_+ G^{\mathbb{C}} \to \Lambda G^{\mathbb{C}}. \tag{3}$$

Consider now the subspace of $\Lambda^\varepsilon \mathfrak{g}$ defined by

$$\Lambda^\varepsilon_{-1,\infty} = \{\xi \in \Lambda^\varepsilon \mathfrak{g} \mid \lambda\xi \text{ extends holomorphically to } I_\varepsilon\};$$

each element $\xi \in \Lambda^\varepsilon_{-1,\infty}$ can be written as $\xi = (\xi_+, \xi_-)$, where $\xi_+ : C_\varepsilon \to \mathfrak{g}^{\mathbb{C}}$ extends meromorphically to I_ε with at most a simple pole at 0 and $\xi_- : C_{1/\varepsilon} \to \mathfrak{g}^{\mathbb{C}}$ is defined by $\xi_-(\lambda) = \overline{\xi_+(1/\bar{\lambda})}$. A $\Lambda^\varepsilon_{-1,\infty}$-valued 1-form $\mu = (\mu_+, \mu_-)$ on $\mathbb{C}$ is called a *ε-holomorphic potential* if μ_+ is holomorphic. We denote by $\mathcal{P}^\varepsilon$ the vector space of all ε-holomorphic potentials.

Remark 2. Recall that, according to [8], a holomorphic 1-form is called a *holomorphic potential* if it has values in $\Lambda_{-1,\infty} = \{\xi \in \Lambda\mathfrak{g} \mid \lambda\xi \text{ extends holomorphically to } I\}$, where $I = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. The vector space of holomorphic potentials is denoted by $\mathcal{P}$. We can interpret the vector space $\mathcal{P}$ as the limiting case of $\mathcal{P}^\varepsilon$ as $\varepsilon \to 1$.

Let μ be a holomorphic ε-potential, so that $d_\mu = d + \mu$ is a flat connection. This means that we can integrate to obtain a smooth map $\Psi_\mu : \mathbb{C} \to \Lambda^\varepsilon G$, which is unique up to left multiplication by a constant loop, with $\Psi_\mu^{-1} d\Psi_\mu = \mu$. We call Ψ_μ a *complex extended solution.* If we factorize Ψ_μ according to (2), we obtain a map $\Phi_\mu : \mathbb{C} \to \Omega^\varepsilon_E G$ such that $\Psi_\mu = \Phi_\mu b$, with $b : \mathbb{C} \to \Lambda^\varepsilon_I G$. It happens that $\Phi_\mu : \mathbb{C} \to \Omega^\varepsilon_E G \subset \Omega G$ is an extended solution (cf. [6, 8]). Then any ε-holomorphic potential μ defines a harmonic map $\phi_\mu : \mathbb{C} \to G$, which is unique up to left multiplication by a constant element, by $\phi_\mu(z) = \Phi_\mu(z)(-1)$. Conversely, any harmonic map from $\mathbb{C}$ to G is obtained in this way from a ε-holomorphic potential.

Write $b_0 = b_{|\lambda=0} : \mathbb{C} \to G^{\mathbb{C}}$ and $\mu_+ = \sum_{i=-1}^{\infty} \lambda^i \mu_i$. The coefficient in λ^{-1} of $\Phi_\mu d\Phi_\mu$ is given by $b_0 \mu_{-1} b_0^{-1}$. Hence, if $\mu_{-1} = 0$, the harmonic map ϕ_μ is constant.

With respect to the limiting case as $\varepsilon \to 1$, consider a potential of the form $\mu = \lambda^{-1}\eta$, with η a meromorphic function. This potential gives rise to a harmonic map ϕ_μ, well defined off a discrete set of points. Sometimes these singularities are removable and ϕ_μ can be extended to all $\mathbb{C}$. In [8], the authors observed that any harmonic map from $\mathbb{C}$ to G can be obtained, up to left multiplication by a constant, from a potential of the form $\mu = \lambda^{-1}\eta$, if we allow η to be meromorphic.

2.3 Harmonic Maps of Finite Uniton Number

A harmonic map $\phi : \mathbb{C} \to G$ is called a *harmonic map of finite uniton number* if it admits an extended solution $\Phi : \mathbb{C} \to \Lambda G^{\mathbb{C}}$ with Fourier series with finitely many terms. For example, it is well known that all harmonic two-spheres in G have finite uniton number. Let us recall which meromorphic potentials give rise to this kind of harmonic maps, referring the reader to [4, 9] for details.

First recall that there is some freedom in the choice of a complex extended solution for a certain harmonic map. In fact:

Proposition 1. [9] *Let $\Psi : \mathbb{C} \to \Lambda G^{\mathbb{C}}$ be a complex extended solution. Let $\Upsilon : \mathbb{C} \to \Lambda_+ G^{\mathbb{C}}$ be a holomorphic map. Then the product $\Psi\Upsilon$ is also a complex extended solution (associated to the same harmonic map).*

Now, let G be a compact semisimple Lie group and choose a maximal torus T of G. Denote by $\mathfrak{t}$ and $\mathfrak{g}$ the Lie algebras of T and G, respectively. Let Δ be the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$: thus $\Delta \subset \sqrt{-1}\mathfrak{t}^*$. For each root $\alpha \in \Delta$ denote by $\mathfrak{g}^\alpha$ the corresponding root space. Fix a positive root system Δ^+ with simple roots $\alpha_1, \ldots, \alpha_l$ and a subset I of $\{1, \ldots, l\}$. Define the "height" function n_I on Δ by $n_I(\alpha) = \sum_{i \in I} n_i$, for $\alpha = \sum_{i=1}^{l} n_i \alpha_i$. Hence

$$\mathfrak{p}_I = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{n_I(\alpha) \geq 0} \mathfrak{g}^\alpha$$

is a parabolic subalgebra. Set

$$\mathfrak{g}_j = \sum_{n_I(\alpha) = j} \mathfrak{g}^\alpha.$$

Hence $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and

$$\mathfrak{g}^{\mathbb{C}} = \sum_{j=-k}^{k} \mathfrak{g}_j.$$

Since $\mathfrak{g}^{\mathbb{C}}$ is semisimple (and so every derivation is an inner derivation), we conclude that there is a unique $\xi_I \in \mathfrak{g}^{\mathbb{C}}$ with $\mathrm{ad}\xi_I = j\sqrt{-1}$ on $\mathfrak{g}_j$ for all $j \in \{-k, \ldots, k\}$. The element ξ_I is called the *canonical element* of $\mathfrak{p}_I$. Observe that $\mathrm{ad}\xi_I$ has values in $\mathfrak{g}$ when restricted to $\mathfrak{g}$. On the other hand, the compact real Lie algebra $\mathfrak{g}$, being semisimple, has trivial center $\mathfrak{z}(\mathfrak{g}) = \{0\}$. Whence $\xi_I \in \mathfrak{g}$. At same time, ξ_I centralizes $\mathfrak{h} = \mathfrak{p}_I \cap \bar{\mathfrak{p}}_I \cap \mathfrak{g}$. So ξ_I belongs to the centre of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{z}(\mathfrak{h}) \subset \mathfrak{g}$.

Let $\xi_1, \ldots, \xi_l \in \mathfrak{t}$ be dual to $\alpha_1, \ldots, \alpha_l$, in the sense that $\alpha_i(\xi_j) = \delta_{ij}\sqrt{-1}$. Then $\xi_I = \sum_{i \in I} \xi_i$. Define

$$\mathfrak{p}_I^{(i)} = \bigoplus_{j \geq i} \mathfrak{g}_j = \bigoplus_{n_I(\alpha) \geq i} \mathfrak{g}^\alpha.$$

Theorem 1. [4, 9] *Let Ψ be a complex extended solution, corresponding to a harmonic map $\phi : \mathbb{C} \to G$ of finite uniton number. Then there exists a parabolic subalgebra $\mathfrak{p}_I$ and a gauge transformation (in the sense of Proposition 1) which converts Ψ to the canonical form $\Psi(z)(\lambda) = \exp B(z, \lambda)$, where*

$$B(z, \lambda) = \lambda^{-1} B_1(z) + \lambda^{-2} B_2(z) + \ldots + \lambda^{-k} B_k(z) \tag{4}$$

for some $B_1, \ldots, B_k$, with B_i a $\mathfrak{p}_I^{(i)}$-valued meromorphic function.

Conversely, let $B_1 : \mathbb{C} \to \mathfrak{p}_I^{(1)}$ be any meromorphic function. Let $B_2, \ldots, B_k$, with B_i a $\mathfrak{p}_I^{(i)}$-valued meromorphic function, obtained by solving recursively the system of meromorphic differential equations

$$(\exp B)^{-1} (\exp B)' = \lambda^{-1} B_1',$$

with B of the form (4). Then $\Psi = \exp B$ is a complex extended solution (corresponding to a harmonic map $\phi : \mathbb{C} \to G$ of finite uniton number).

Hence, the meromorphic potential $\mu = \lambda^{-1} B_1' dz$ on $\mathbb{C}$ gives rise to a harmonic map of finite uniton number with complex extended solution $\Psi_\mu = \exp B$.

Remark 3. Suppose that the Fourier series associated to the complex extended solution $\Psi : \mathbb{C} \to \Lambda G^{\mathbb{C}} \subset \Lambda \mathrm{Gl}(n, \mathbb{C})$ has finitely many terms: $\Psi = \sum_{i=-k}^{k} A_i \lambda^i$. Let $\phi : \mathbb{C} \to G$ be the corresponding harmonic map. It is well known [10, 17] that, in this case, Ψ corresponds, via the Grassmannian model for loop groups [15], to a holomorphic vector subbundle W of the finite dimensional trivial bundle over $\mathbb{C}$ given by

$$\underline{E} = \bigoplus_{i=-k}^{k} \lambda^i \underline{\mathbb{C}}^n.$$

If Ψ is meromorphic on $\mathbb{C}$, W admits a frame field formed by meromorphic sections $s_1, \ldots, s_r$ of $\underline{E}$ over $\mathbb{C}$. Hence $s_1 \wedge \ldots \wedge s_r$ is a meromorphic section of $\bigwedge^r \underline{E}$. Suppose that $s_1 \wedge \ldots \wedge s_r$ has a pole of order k_0 at z_0. Then $s_1 \wedge \ldots \wedge s_r = (z - z_0)^{-k_0} S$ where S is a local holomorphic section of $\bigwedge^r \underline{E}$ such that $S(z_0) \neq 0$ and S is decomposible for all z near z_0. Thus S defines a local rank r holomorphic subbundle which coincides with W off z_0. This means that W can be holomorphically extended to all $\mathbb{C}$, that is, the singularities of ϕ are removable. Observe that if Ψ is in the canonical form, $\Psi = \exp B$, then the Fourier series of Ψ will have finitely many terms and Ψ will be meromorphic, since B is nilpotent and meromorphic. Hence the harmonic map associated to $\Psi = \exp B$ admits an extension to all $\mathbb{C}$.

3 Dressing Actions and Singular Dressing Actions

The Iwasawa-type decomposition (2) allows us to define a natural action $\#_\varepsilon$ of $\Lambda_I^\varepsilon G$ on $\Omega_E^\varepsilon G$: if $g \in \Omega_E^\varepsilon G$ and $h \in \Lambda_I^\varepsilon G$, then $h \#_\varepsilon g = (hg)_E$. Applying this action pointwise, we obtain from an extended solution $\Phi : \mathbb{C} \to \Omega G$ a new extented solution $h\#_\varepsilon \Phi : \mathbb{C} \to \Omega G$ defined by $(h\#_\varepsilon \Phi)(z) = h\#_\varepsilon \Phi(z)$, for all $z \in \mathbb{C}$ [5, 8, 11]. As explained in [5], the action of each $\Lambda_I^\varepsilon G$ preserves

$$\Omega_{\mathrm{hol}} G = \big\{\gamma : \mathbb{C}^* \to G^{\mathbb{C}} \mid \gamma \text{ is holomorphic}, \gamma(1) = e \text{ and } \overline{\gamma(\lambda)} = \gamma\big(1/\bar{\lambda}\big)\big\};$$

and for $0 < \varepsilon < \varepsilon' < 1$, $g \in \Lambda_I^{\varepsilon'} G \subset \Lambda_I^\varepsilon G$, and $h \in \Lambda_{\mathrm{hol}} G$, we have $g\#_{\varepsilon'} h = g\#_\varepsilon h$. Hence, by taking a direct limit as $\varepsilon \to 0$, we get an action of $\mathcal{G}_0$, the group of germs at zero of holomorphic maps $\mathbb{C} \to G^{\mathbb{C}}$, on $\Omega_{\mathrm{hol}} G$. Henceforth, we write $\gamma\# g$ for this action on $\Omega_{\mathrm{hol}} G$.

On the other hand, the holomorphic gauge group

$$\mathcal{G}^\varepsilon = \big\{h = (h_+, h_-) : \mathbb{C} \to \Lambda_I^\varepsilon G \ \text{ such that } \bar{\partial} h_+ = 0\big\}$$

acts on the space $\mathcal{P}^\varepsilon$ of ε-holomorphic potentials by gauge transformations: if $\mu \in \mathcal{P}^\varepsilon$ and $h \in \Lambda_I^\varepsilon G$, then $h \cdot \mu = \mathrm{Ad}_h(\mu) - dh\, h^{-1} \in \mathcal{P}^\varepsilon$. It happens that the correspondence between holomorphic potentials and extended solutions $\mu \to \Phi_\mu$ is equivariant with respect to these actions [5, 8], that is, if $h \in \Lambda_I^\varepsilon G$, then

$$\Phi_{h\cdot\mu} = h(0)\#\Phi_\mu. \tag{5}$$

Remark 4. The limiting case as $\varepsilon \to 1$ can be stated as follows: consider the dressing action $\#$ of $\Lambda_+ G^{\mathbb{C}}$ on ΩG corresponding to the Iwasawa decomposition of Remark 1; let $\mathcal{G}$ be the holomorphic gauge group of all $h : \mathbb{C} \to \Lambda_+ G^{\mathbb{C}}$ such that $\bar{\partial} h = 0$; let $\mu \in \mathcal{P}$ be a holomorphic potential; then $\Phi_{h\cdot\mu} = h(0)\#\Phi_\mu$.

Suppose that $\phi : \mathbb{C} \to G$ is a harmonic map of finite uniton number with finite energy: $\int_{\mathbb{C}} |d\phi|^2 < \infty$. It is well known [16] that, in this case, ϕ extends to a smooth harmonic map on S^2. On the other hand, in [1] the authors proved that the dressing action preserves the energy. Hence any new harmonic map $\tilde{\phi}$ obtained from ϕ by dressing also admits an extension to S^2.

3.1 Singular Dressing Actions

Consider now a curve $\{\gamma_a\}$ in $\mathcal{G}_0$ and $\Phi : \mathbb{C} \to \Omega G$ an extended solution. Suppose that $\tilde{\Phi} = \lim_{a\to 0} \left(\gamma_a \# \Phi\right)$ has a finite number of removable singularities. In this case, we obtain a new extended solution $\tilde{\Phi}$ by removing these singularities. This procedure of obtaining $\tilde{\Phi}$ from Φ is called *modified completion procedure* or the *singular dressing action* [1, 12].

In [1], the authors proved that any harmonic map $\mathrm{S}^2 \to \mathbb{C}\mathrm{P}^n$ may be reduced to a constant by applying twice the singular dressing action procedure. Jiao [12] generalized this result and proved that any harmonic map $\mathrm{S}^2 \to \mathrm{U}(n)$ may be reduced to a constant by applying n singular dressing actions. In both cases, the singular dressing actions are produced from curves $\{\gamma_a\}$ of rational loops of the form

$$\gamma_a(\lambda) = \pi_V^{\perp} + \zeta_a(\lambda)\pi_V, \tag{6}$$

where $\zeta_a(\lambda)$ is given by (1). The rational loops (6) are precisely the *simple factors* [18] for the action of the group of germs at zero of holomorphic maps from $\mathbb{C}$ to $\mathrm{Gl}(n, \mathbb{C})$ on $\Omega_{\mathrm{hol}}\mathrm{U}(n)$. Their dressing action is explicitly calculable and they generate the group of rational loops in $\mathrm{Gl}(n, \mathbb{C})$ satisfying the reality condition $\overline{\gamma(\lambda)} = \gamma\left(1/\bar{\lambda}\right)$. Moreover, for suitable complex subspace V of $\mathbb{C}^n$, the singular dressing action $\tilde{\Phi} = \lim_{a\to 0} \left(\gamma_a \# \Phi\right)$ amounts to adding a uniton to Φ [1, 18]. However, as observed in [1], not all unitons can be added via singular dressing actions.

In Section 4 we will prove that that any harmonic map ϕ from S^2 into an arbitrary compact semisimple matrix Lie group G may be reduced to a constant by applying a finite number of singular dressing actions; this reduction induces a factorization of ϕ into flag factors $\mathrm{S}^2 \to \mathrm{G}$, and the singular dressing actions are produced from curves of simple factors for $G^{\mathbb{C}}$. But first we have to define our simple factors (inspired by [3, 7]):

Definition 1. Let G be a compact semisimple matrix Lie group. For any $a \in \mathbb{C}^*$ and ξ in the integer lattice $\mathfrak{I} = (2\pi)^{-1}\exp^{-1}(e) \cap \mathfrak{g}$, the loop

$$p_{a,\xi}(\lambda) = \exp\Big(\ln\Big(\zeta_a(\lambda)\Big)\sqrt{-1}\xi\Big), \tag{7}$$

where $\zeta_a(\lambda)$ is given by (1), is a *simple factor*.

The condition $\xi \in \mathfrak{I}$ ensures that $p_{a,\xi}$ is well defined and satisfies the reality condition $\overline{p_{a,\xi}(\lambda)} = p_{a,\xi}(1/\overline{\lambda})$. Moreover, one can easily check that $p_{a,\xi}$ is rational.

Definition 2. If $p_{a,\xi}$ is a simple factor, the singular dressing action $\tilde{\Phi} = \lim_{a\to 0}\left(p_{a,\xi}\#\Phi\right)$ is a *simple singular dressing action.*

3.2 Generators for Classical Rational Loop Groups

Let G be a matrix Lie group and $\Omega_{\mathrm{rat}}G$ be the group of rational loops in $G^{\mathbb{C}}$ satisfying the reality condition $\overline{\gamma(\lambda)} = \gamma\left(1/\bar{\lambda}\right)$ and $\gamma(1) = e$. This group acts by dressing on the space of harmonic maps into G. Hence, it is important to know the generators of $\Omega_{\mathrm{rat}}G$ and how to compute their action. In [7] the authors proved that $\Omega_{\mathrm{rat}}G$ with $G = \mathrm{SO}(n)$ is generated by simple factors of the form (7). However, as we shall show next, in $\mathrm{Sp}(n)$ and $\mathrm{SU}(n)$ cases one has to enlarge the class of generators.

The $\mathrm{Sp}(n)$ *and* $\mathrm{SO}(n)$ *cases.* Let $\langle\cdot,\cdot\rangle$ be a hermitian inner product on a n-dimensional complex vector space V. Suppose that V carries a j-structure, that is, a $\mathbb{R}$-linear map $j : V \to V$ satisfying: $j^2 = \pm 1$; $j(zv) = \bar{z}j(v)$ for all $z \in \mathbb{C}$ and $v \in V$; and $\langle jv, jw\rangle = \overline{\langle v, w\rangle}$. Define the bilinear map $\Omega : V^2 \to \mathbb{C}$ by $\Omega(v,w) = \langle v, jw\rangle$. Let $G \subset \mathrm{U}(n)$ be the compact semisimple Lie group that preserves Ω. Observe that Ω is symmetric if $j^2 = 1$ (the $\mathrm{SO}(n)$-case) and Ω is anti-symmetric if $j^2 = -1$ (the $\mathrm{Sp}(n)$-case). Recall that a subspace L of V is said to be j-isotropic if jL is orthogonal to L.

Lemma 1. *Consider a loop* $\gamma \in \Lambda\mathrm{U}(n)$ *of the form*

$$\gamma(\lambda) = \left(\pi_L\lambda^{-1} + \pi_L^\perp\right)\left(\pi_W\lambda + \pi_W^\perp\right)$$

with $\dim L = \dim W = 1$. *Then* γ *is in* ΛG *if and only if* L *and* W *are* j*-isotropic and* $W \subset L \oplus jL$ *(equivalently,* $L \subset W \oplus jW$*). If* $j^2 = 1$*, then either* $W = L$ *or* $W = jL$*.*

Proof. If $\gamma \in \Lambda G$, then $\gamma(\lambda) = \pm j\gamma(\lambda)j$ for all $\lambda \in \mathbb{C}^*$, that is,

$$\begin{aligned}
\big(\pi_L\lambda^{-1} + \pi_L^{\perp}\big)\big(\pi_W\lambda + \pi_W^{\perp}\big) &= \pm j\big(\pi_L\lambda^{-1} + \pi_L^{\perp}\big)\big(\pi_W\lambda + \pi_W^{\perp}\big)j \\
&= j\big(\pi_L\lambda^{-1} + \pi_L^{\perp}\big)jj\big(\pi_W\lambda + \pi_W^{\perp}\big)j \\
&= \big(\pi_{jL}\lambda + \pi_{jL}^{\perp}\big)\big(\pi_{jW}\lambda^{-1} + \pi_{jW}^{\perp}\big).
\end{aligned}$$

Comparing coefficients of λ, we see that:

$$\pi_L\pi_W^{\perp} = \pi_{jL}^{\perp}\pi_{jW}, \quad \pi_L\pi_W + \pi_L^{\perp}\pi_W^{\perp} = \pi_{jL}\pi_{jW} + \pi_{jL}^{\perp}\pi_{jW}^{\perp}.$$

The first of these equations implies, in the non-trivial case $L \neq W$, that L and W are j-isotropic. On the other hand, if $w \in W$, the second equation implies that $\pi_L(w) = \pi_{jL}^{\perp}(w)$. Since L is j-isotropic, this means that $W \subset L \oplus jL$

Suppose now that $j^2 = 1$. Take $w_1 + jw_2 \in W$ with $w_1, w_2 \in L$. Since L and W are j-isotropic, we get

$$\begin{aligned}
0 &= \Omega(w_1 + jw_2, w_1 + jw_2) = \langle w_1 + jw_2, jw_1 + w_2\rangle \\
&= \langle w_1, w_2\rangle + \langle jw_2, jw_1\rangle = \langle w_1, w_2\rangle + \overline{\langle w_2, w_1\rangle} \\
&= 2\langle w_1, w_2\rangle.
\end{aligned}$$

Hence, either $w_1 = 0$ or $w_2 = 0$, which means that either $W = L$ or $W = jL$. □

In [7], the authors have proved that the group $\Omega_{\mathrm{rat}}\mathrm{SO}(n)$ is generated by simple factors. In the $\mathrm{Sp}(n)$-case the class of generators is slightly larger:

Theorem 2. *The group of rational loops $\Omega_{\mathrm{rat}}G$ is generated by loops of the form*

$$q_{a,L,W}(\lambda) = \big(\pi_L\zeta_a(\lambda)^{-1} + \pi_L^{\perp}\big)\big(\pi_W\zeta_a(\lambda) + \pi_W^{\perp}\big) \tag{8}$$

with L and W two j-isotropic lines such that $L \subset W \oplus jW$, and $a \in \mathbb{C} \setminus \{S^1\}$.

Proof. Our proof is inspired in [7]. Take $\Phi \in \Omega_{\mathrm{rat}}G$. Use the linear fractional $\zeta_a : \mathbb{P}^1 \to \mathbb{P}^1$ defined by (1) to move the singularities at a and $1/\bar{a}$ to 0 and ∞, with $a \in \mathbb{C} \setminus \{\mathrm{S}^1\}$. Consider the Laurent series of $\Phi \circ \zeta_a^{-1}$ in powers of λ:

$$\Phi \circ \zeta_a^{-1}(\lambda) = \lambda^{-k}\Phi_{-k} + \lambda^{-k+1}\Phi_{-k+1} + \lambda^{-k+2}\Phi_{-k+2} + \dots$$

with $\Phi_{-k} \neq 0$. Since Φ takes values in $G^{\mathbb{C}}$, it preserves the bilinear form Ω, that is,

$$\Omega\big(\Phi \circ \zeta_a^{-1}(\lambda)v, \Phi \circ \zeta_a^{-1}(\lambda)w\big) = \Omega(v,w)$$

for all $v, w \in V$ and all $\lambda \neq 0, \infty$. Comparing coefficients of λ we see that

$$\Omega(\Phi_{-k}v, \Phi_{-k}w) = 0 \tag{9}$$

$$\Omega(\Phi_{-k+1}v, \Phi_{-k}w) + \Omega(\Phi_{-k}v, \Phi_{-k+1}w) = 0. \tag{10}$$

Equation (9) says that $\mathrm{Im}\Phi_{-k}$ is j-isotropic. Hence:

$$V = \mathrm{Im}\Phi_{-k} \oplus \{\mathrm{Im}\Phi_{-k} \oplus j\mathrm{Im}\Phi_{-k}\}^{\perp} \oplus j\mathrm{Im}\Phi_{-k}.$$

Now, let $W \subset \mathrm{Im}\Phi_{-k}$ be a line. Let L be another j-isotropic line. By Lemma 1, the rational loop $q_{a,L,W}(\lambda)$ takes values in $G^{\mathbb{C}}$ if and only if $L \subset W \oplus jW$. Observe that in $j^2 = 1$ case there are only two possibilities: $L = W$ or $L = jW$. We have:

$$\begin{aligned}
\sum_{i=-k-1}^{\infty} \lambda^i \Psi_i \;&:=\; \big(q_{a,L,W}\Phi\big) \circ \zeta_a^{-1}(\lambda) \\
&= \lambda^{-k-1}\pi_L \pi_{jW}\Phi_{-k} + \lambda^{-k}\Big(\pi_L^{\perp}\pi_W^{\perp}\Phi_{-k} + \pi_L\pi_W\Phi_{-k} + \pi_L\pi_{jW}\Phi_{-k+1}\Big) + \dots
\end{aligned}$$

Since $\mathrm{Im}\Phi_{-k}$ is isotropic we have

$$\ker \pi_{jW} = jW^{\perp} \supset j\mathrm{Im}\Phi_{-k}^{\perp} \supset \mathrm{Im}\Phi_{-k},$$

hence Ψ_{-k-1} vanishes, that is, the multiplication on the left by $q_{a,L,W}$ does not increase the order of the pole of Φ at a. For a suitable choice of L the rank of Ψ_{-k} is smaller than the rank of Φ_{-k}. In fact:

Take $v \in \ker \Phi_{-k}$. Then, by (10), we have $\Omega(\Phi_{-k+1}v, \Phi_{-k}w) = 0$ for all $w \in V$. Thus

$$\Phi_{-k+1}(\ker \Phi_{-k}) \perp j\mathrm{Im}\Phi_{-k} \supset jW.$$

In particular, we have $\ker \Phi_{-k} \subset \ker \Psi_{-k}$. Let $v_0 \in V$ such that $\Phi_{-k}v_0 \in W \setminus \{0\}$ and choose L to be the line generated by

$$j\big(\Phi_{-k}v_0 + \pi_{jW}\Phi_{-k+1}v_0\big).$$

When $j^2 = 1$, if we take $v, w = v_0$ in (10) and use the symmetry of Ω, then we conclude that $L = JW$, and consequently L is isotropic; when $j^2 = -1$, the isotropy of L follows directly from the fact that L is one-dimensional. With this choice, the inclusion $\ker \Phi_{-k} \subset \ker \Psi_{-k}$ is proper, which means that $\mathrm{rank}\Psi_{-k} < \mathrm{rank}\Phi_{-k}$.

We can continue this process until remove the singularity at a. By the reality condition, the singularity at $1/\bar{a}$ is simultaneously removed. Hence, since Φ has a finite number of singularities, we obtain from Φ a holomorphic map $\mathbb{P}^1 \to G^{\mathbb{C}}$ after multiplication on the left by a finite number of loops of the form (8). Since the only holomorphic maps on compacts are the constants, we conclude that the loops of the form (8) generate $\Omega_{\mathrm{rat}}G$.

$\square$

Remark 5. The rational loop $q_{a,L,W}$ is a non-trivial simple factor, in the sense of Definition 1, if and only if $W = jL$. Hence, we conclude with [7], Theorem 5.1, that the group of rational loops in $\mathrm{SO}(n)$ is generated by the simple factors $q_{a,L,jL}$. When $W = jL$ we denote $q_{a,L} = q_{a,L,jL}$.

In order to compute the dressing action of these generators, one can apply twice the well known [18] formula for the dressing action of the simple factors (6) on $\Omega_{\mathrm{hol}}\mathrm{U}(n)$. Explicitly:

Let H_L be the element of $\mathfrak{u}(n)$ given by $\sqrt{-1}$ on L and 0 on $L^\perp$. Similarly, define $H_W \in \mathfrak{u}(n)$. We have $q_{a,L,W} = p^{-1}_{a,H_L} p_{a,H_W}$ and

$$q_{a,L,W} \# \Phi = p^{-1}_{a,H_L} \# (p_{a,H_W} \# \Phi) = q_{a,L,W} \Phi q^{-1}_{a,\hat{L},\hat{W}} \in \Omega_{\mathrm{hol}} G \subset \Omega_{\mathrm{hol}} \mathrm{U}(n),$$

where $\hat{W} = \Phi(a)^{-1} W$ and $\hat{L} = (p_{a,H_W} \# \Phi)(a) L$. Hence, Lemma 1 ensures that the lines $\hat{L}$ and $\hat{W}$ are j-isotropic and $\hat{L} \subset \hat{W} \oplus j\hat{W}$. When $j^2 = 1$, we must have, in the non-trivial case, $W = jL$ and $\hat{W} = j\hat{L}$, and we recover the well known formula [3, 7] for the dressing action of $q_{a,L}$:

$$q_{a,L} \# \Phi = q_{a,L} \Phi q^{-1}_{a,\hat{L}},$$

with $\hat{L} = \Phi(a)^{-1} L$.

The $\mathrm{SU}(n)$ *case.* Observe that a loop $\gamma \in \Lambda \mathrm{U}(n)$ of the form

$$\gamma(\lambda) = \left(\pi_L \lambda^{-1} + \pi_L^\perp\right)\left(\pi_W \lambda + \pi_W^\perp\right)$$

is in $\Lambda\mathrm{SU}(n)$ if and only if $\dim L = \dim W$. We have:

Theorem 3. *The group of rational loops* $\Omega_{\mathrm{rat}}\mathrm{SU}(n)$ *is generated by loops of the form*

$$q_{a,L,W}(\lambda) = \left(\pi_L \zeta_a(\lambda)^{-1} + \pi_L^\perp\right)\left(\pi_W \zeta_a(\lambda) + \pi_W^\perp\right)$$

with L *and* W *subspaces such that* $\dim L = \dim W$*, and* $a \in \mathbb{C} \setminus \{\mathrm{S}^1\}$.

Proof. Let $\Phi : \mathbb{P}^1 \to \mathrm{Sl}(n, \mathbb{C})$ be a rational loop satisfying the reality condition. Fix a pole $a \in \mathbb{C} \setminus \{\mathrm{S}^1\}$. Again, we use the linear fractional ζ_a to move the singularities at a and $1/\bar{a}$ to 0 and ∞. We write the Laurent series of $\Phi \circ \zeta_a^{-1}$ in λ explicitly as

$$\Phi \circ \zeta_a^{-1}(\lambda) = \lambda^{-k}\Phi_{-k} + \lambda^{-k+1}\Phi_{-k+1} + \lambda^{-k+2}\Phi_{-k+2} + \dots$$

with $\Phi_{-k} \neq 0$.

Since $\det\{\lambda^k \Phi \circ \zeta_a^{-1}(\lambda)\} = \lambda^{kn}$, evaluating at $\lambda = 0$ we get $\det \Phi_{-k} = 0$. Set $W = \mathrm{Im}\Phi_{-k}$ and $\gamma_{a,W} \circ \zeta_a^{-1}(\lambda) = \lambda \pi_W + \pi_W^{\perp}$. Then

$$\Psi \circ \zeta_a^{-1}(\lambda) = \sum_{i=-k+1}^{\infty} \lambda^i \Psi_i := \left(\gamma_{a,W}\Phi\right) \circ \zeta_a^{-1}(\lambda) = \lambda^{-k+1}\left(\pi_W \Phi_{-k} + \pi_W^{\perp}\Phi_{-k+1}\right) + \dots$$

Observe that

$$\det\{\lambda^{k-1}\Psi \circ \zeta_a^{-1}(\lambda)\} = \lambda^{(k-1)n+m},$$

where $m = \dim W$. Evaluating at $\lambda = 0$ we get $\det\{\pi_W \Phi_{-k} + \pi_W^{\perp}\Phi_{-k+1}\} = 0$. In particular,

$$R := \mathrm{Im}^{\perp}\{\pi_W \Phi_{-k} + \pi_W^{\perp}\Phi_{-k+1}\} \neq \{0\}.$$

Consider a m-dimensional subspace L containing at least one line of R. For such choice, we have $\mathrm{rank}\, \pi_L\left(\pi_W \Phi_{-k} + \pi_W^{\perp}\Phi_{-k+1}\right) < \mathrm{rank}\Phi_{-k}$. Since

$$\left(q_{a,L,W}\Phi\right) \circ \zeta_a^{-1}(\lambda) = \left(\gamma_{a,L}^{-1}\Psi\right) \circ \zeta_a^{-1}(\lambda) = \lambda^{-k}\pi_L\left(\pi_W \Phi_{-k} + \pi_W^{\perp}\Phi_{-k+1}\right) + \dots,$$

we can continue this process, as in the proof of Theorem 2, until remove the singularities at a and $1/\bar{a}$. $\square$

Again, the dressing action of the generators $q_{a,L,W}$ can be computed by applying twice the formula for the dressing action of the simple factors (6).

4 Factorizations by Simple Singular Dressing Actions

Let G be a compact semisimple matrix Lie group and choose a maximal torus T of G. Denote by $\mathfrak{t}$ and $\mathfrak{g}$ the Lie algebras of T and G, respectively. Let Δ be the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. Fix a positive root system Δ^+ with simple roots $\alpha_1, \dots, \alpha_l$ and a subset $I = \{i_1, \dots, i_k\}$ of $\{1, \dots, l\}$. Let $\Psi = \exp B : \mathbb{C} \setminus D \to \Lambda G^{\mathbb{C}}$ be a complex extended solution, D a discrete subset, associated to the meromorphic potential $\mu = \lambda^{-1} B_1' dz$, corresponding to a harmonic map $\phi : \mathbb{C} \to G$ of finite uniton number with extended solution $\Phi : \mathbb{C} \setminus D \to \Omega_{\mathrm{hol}} G$, according to the notations of Theorem 1. In particular, B_1 is a $\mathfrak{p}_I^{(1)}$-valued meromorphic function. We have:

Theorem 4. *The harmonic map ϕ can be reduced to a constant by applying k simple singular dressing actions. Hence, any harmonic map from S^2 to G can be reduced to a constant by applying a finite number of simple singular dressing actions.*

Proof. Let $\xi_1, \ldots, \xi_l \in \mathfrak{t}$ be dual to $\alpha_1, \ldots, \alpha_l$, in the sense that $\alpha_i(\xi_j) = \delta_{ij}\sqrt{-1}$. Given $\alpha \in \Delta$, write $\alpha = \sum_{i=1}^{l} n_i(\alpha)\alpha_i$. Clearly, $\xi_i \in \mathfrak{I}$ and $n_i(\alpha)$ is a non-negative integer number for each $\alpha \in \Delta^+$ and $i \in \{1, \ldots, l\}$.

Consider the simple factor $p_{a,\xi_{i_1}}$, with $0 < |a| < 1$ and its gauge action on the meromorphic potential μ:

$$
\begin{aligned}
\mu^{i_1,a} &:= p_{a,\xi_{i_1}}^{-1} \cdot \mu = \lambda^{-1} \mathrm{Ad}_{\exp\left(-\ln(\zeta_a(\lambda))\sqrt{-1}\xi_{i_1}\right)} \big(B_1' dz\big) \\
&= \lambda^{-1} e^{-\ln(\zeta_a(\lambda))\mathrm{ad}(\sqrt{-1}\xi_{i_1})} \big(B_1' dz\big) \\
&= \lambda^{-1} \sum_{n=0}^{+\infty} \frac{(-\ln \zeta_a(\lambda))^n}{n!} \mathrm{ad}(\sqrt{-1}\xi_{i_1})^n (B_1' dz) \\
&= \lambda^{-1} \sum_{n_I(\alpha)\geq 1} \sum_{n=0}^{+\infty} \frac{(-\ln \zeta_a(\lambda))^n}{n!} \mathrm{ad}(\sqrt{-1}\xi_{i_1})^n (B_{1\alpha}' dz) \\
&= \lambda^{-1} \sum_{n_I(\alpha)\geq 1} \sum_{n=0}^{+\infty} \frac{(-\ln \zeta_a(\lambda))^n}{n!} (-1)^n n_{i_1}(\alpha)^n B_{1\alpha}' dz \\
&= \lambda^{-1} \sum_{\substack{n_I(\alpha)\geq 1 \\ n_{i_1}(\alpha)=0}} B_{1\alpha}' dz + \lambda^{-1} \sum_{\substack{n_I(\alpha)\geq 1 \\ n_{i_1}(\alpha)>0}} \sum_{n=0}^{+\infty} \frac{(-\ln \zeta_a(\lambda))^n}{n!} (-1)^n n_{i_1}(\alpha)^n B_{1\alpha}' dz \\
&= \lambda^{-1} \sum_{\substack{n_I(\alpha)\geq 1 \\ n_{i_1}(\alpha)=0}} B_{1\alpha}' dz + \lambda^{-1} \zeta_a(\lambda)^{n_{i_1}(\alpha)} \sum_{\substack{n_I(\alpha)\geq 1 \\ n_{i_1}(\alpha)>0}} B_{1\alpha}' dz
\end{aligned}
$$

Hence, although p_{a,ξ_1} does not belong to $\Lambda_+ G^{\mathbb{C}}$, the new potential $\mu^{i_1,a}$ extends meromorphically in λ to the unitary disc $I = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ with at most a simple pole at 0. This potential corresponds to the complex extended solution

$$\Psi^{i_1,a} = \mathrm{Ad}_{p_{a,\xi_{i_1}}^{-1}} \big(\Psi\big) : \mathbb{C} \setminus D \to \Lambda G^{\mathbb{C}}.$$

If we factorize $\Psi^{i_1,a}$ according to (3), we obtain an extended solution

$$\Phi^{i_1,a} : \mathbb{C} \setminus D \to \Omega G.$$

We observe that $\Psi^{i_1,a}$ can be seen as a map into $\Lambda^\varepsilon G$ for each $0<\varepsilon<|a|$: on C_ε this map coincides with $\Psi^{i_1,a}$; on $C_{1/\varepsilon}$ is determined by the reality condition. Similarly, $\mu^{i_1,a}$ can be seen as a ε-holomorphic potential for each $0<\varepsilon<|a|$. Hence, by (5), $\Phi^{i_1,a}=p^{-1}_{a,\xi_{i_1}}\#\Phi$.

Now, it is clear that $\lim_{a\to 0}\Psi^{i_1,a}=\Psi^{i_1}$ in $\Lambda G^{\mathbb{C}}$, where

$$\Psi^{i_1}=\exp B^{i_1}:\mathbb{C}\setminus D\to\Lambda^\varepsilon G$$

and

$$B^{i_1}=\mathrm{Ad}_{\exp\left(-\ln(\lambda)\sqrt{-1}\xi_{i_1}\right)}(B).$$

The complex extended solution Ψ^{i_1} integrates the potential

$$\mu^{i_1}:=\lim_{a\to 0}\mu^{i_1,a}=\lambda^{-1}\sum_{\substack{n_I(\alpha)\geq 1\\ n_{i_1}(\alpha)=0}}B'_{1\alpha}dz+\lambda^{n_{i_1}(\alpha)-1}\sum_{\substack{n_I(\alpha)\geq 1\\ n_{i_1}(\alpha)>0}}B'_{1\alpha}dz.$$

Again, we can factorize according to (3) in order to obtain an extended solution $\Phi^{i_1}:\mathbb{C}\setminus D\to\Omega G$. By the continuity of decomposition (3), we have $\Phi^{i_1}=\lim_{a\to 0}\Phi^{i_1,a}$. On the other hand, since B^{i_1} is meromorphic and nilpotent, by Remark 3 the corresponding harmonic map ϕ^{i_1} can be extended to all $\mathbb{C}$. Hence, $\phi^{i_1}:\mathbb{C}\to G$ is a harmonic map obtained from ϕ by simple singular dressing action.

If we apply to ϕ^{i_1} the simple singular dressing action defined by $p^{-1}_{a,\xi_{i_2}}$, we obtain a new meromorphic potential

$$\mu^{i_1i_2}=\lambda^{-1}\mu^{i_1i_2}_{-1}+\mu^{i_1i_2}_0+\lambda\mu^{i_1i_2}_1+\dots$$

with

$$\mu^{i_1i_2}_{-1}=\sum_{\substack{n_I(\alpha)\geq 1\\ n_{i_1}(\alpha)=n_{i_2}(\alpha)=0}}B'_{1\alpha}dz.$$

The meromorphic potential $\mu^{i_1i_2}$ gives rise to a complex extended solution $\Psi^{i_1i_2}$ well defined away the singularities. Moreover: $\Psi^{i_1i_2}=\exp B^{i_1i_2}$ with

$$B^{i_1i_2}=\mathrm{Ad}_{\exp\left(-\ln(\lambda)\sqrt{-1}\xi_{i_2}\right)}(B^{i_1}).$$

Again, $B^{i_1i_2}$ is meromorphic and nilpotent. Hence, $\Psi^{i_1i_2}$ corresponds to a harmonic map $\phi^{i_1i_2}$ well defined in all $\mathbb{C}$. If we continue with this procedure, we will end up with a potential $\mu^{i_1i_2\dots i_k}$ with the coefficient associated to λ^{-1} equal to zero. Hence the corresponding harmonic map $\phi^{i_1i_2\dots i_k}$ is constant.

It is clear that if $\phi : \mathbb{C} \to G$ admits an smooth extension to S^2, then each $\phi^{i_1 i_2 \cdots i_k}$ can also be extended to S^2. Since every harmonic map from S^2 to G have finite uniton number, every harmonic map from S^2 to G can be reduced to a constant by applying a finite number of simple singular dressing actions. $\square$

If G is a classical compact semisimple Lie group, this factorization can be refined into a factorization by linear factors. In fact, let $\mathfrak{g} \subset \mathfrak{u}(n)$ be a classical compact semisimple Lie algebra. Choose a maximal torus $\mathfrak{t}$. The subalgebra $\mathfrak{t}$ is generated by elements of the form $H = H^+ + H^-$, where: $H^\pm \in \mathfrak{u}(n)$; H^+ acts diagonally on $\mathbb{C}^n$ with eigenvalues $0, \sqrt{-1}$ (denote by V_0^+, V_1^+ the corresponding eigenspaces); $H^- \in \mathfrak{u}(n)$ acts diagonally on $\mathbb{C}^n$ with eigenvalues $0, -\sqrt{-1}$ (denote by V_0^-, V_1^- the corresponding eigenspaces); $\dim V_1^+ = \dim V_1^- = 1$, and V_1^+ is orthogonal to V_1^-. In particular, $H \in \mathfrak{I}$ and we have:

$$p_{a,H}(\lambda) = \big(\pi_{V_1^+}\zeta_a(\lambda) + \pi^\perp_{V_1^+}\big)\big(\pi_{V_1^-}\zeta_a(\lambda)^{-1} + \pi^\perp_{V_1^-}\big).$$

Since each $p_{a,\xi_{i_k}}$ is a product of rational loops $p_{a,H}$, we conclude that the factorization can be refined into a factorization by linear factors – those corresponding to the singular dressing actions obtained from $\gamma_a(\lambda) = \pi_{V_1^\pm}\zeta_a(\lambda) + \pi^\perp_{V_1^\pm}$.

4.1 Factorizations of Harmonic Maps into Symmetric Spaces

Consider a harmonic map of finite uniton number ϕ from $\mathbb{C}$ into a symmetric space G/K. Following Uhlenbeck [18], consider the Cartan immersion of G/K into G to identify ϕ with a certain harmonic map into G. Let $\Phi : \mathbb{C} \to \Omega G$ be the corresponding extended solution. By Theorem 4, we can reduce ϕ to a constant by applying a finite number of simple singular dressing actions. However, in general, neither $p_{a,\xi}\#\Phi$ nor $\lim_{a\to 0} p_{a,\xi}\#\Phi$ correspond to harmonic maps into symmetric G-spaces. Next we describe how to reduce ϕ to a constant by applying a finite number of singular dresssing actions preserving the symmetry. We shall only consider *inner* symmetric G-spaces.

It is well known [4] that any compact connected inner symmetric space N may be immersed in a certain compact connected Lie group G as a connected component of $\sqrt{e} = \{g \in G : \ g^2 = e\}$. This immersion is totally geodesic. Hence harmonic maps into inner symmetric spaces may be viewed as special harmonic maps into Lie groups. The corresponding extended solutions can be characterized as follows:

Theorem 5. [4, 17, 18] *Consider the involution T on ΩG defined by $T(\gamma)(\lambda) = \gamma(-\lambda)\gamma(-1)^{-1}$. Set $\Omega_T G = \{\gamma \in \Omega G : T(\gamma) = \gamma\}$. Let $\Phi : \mathbb{C} \to \Omega_T G$ be an extended solution. Then $\phi(z) = \Phi(z)(-1)$ defines a harmonic map from $\mathbb{C}$ into $\sqrt{e}$. Conversely, let $\phi : \mathbb{C} \to \sqrt{e}$ be a harmonic map. Assume that ϕ admits an extended solution defined on $\mathbb{C}$. Then there exists an extended solution $\Phi : \mathbb{C} \to \Omega_T G$ such that $\phi(z) = \Phi(z)(-1)$.*

The following lemma establishes which elements of $\mathcal{G}_0$, the group of germs at zero of holomorphic maps $\mathbb{C} \to G^{\mathbb{C}}$, preserve $\Omega_T G$ under dressing.

Lemma 2. *Let $\Phi \in \Omega_T G$ and $\gamma \in \mathcal{G}_0$ such that $\gamma(\lambda) = \gamma(-\lambda)$. Then $\gamma \# \Phi \in \Omega_T G$.*

Proof. We have:

$$\begin{aligned}
\underbrace{T\big((\gamma\Phi)_E\big)(\lambda)}_{\in \Omega_E^{\varepsilon} G} &= (\gamma\Phi)_E(-\lambda)(\gamma\Phi)_E(-1)^{-1} \\
&= (\gamma\Phi)(-\lambda)(\gamma\Phi)_I(-\lambda)^{-1}(\gamma\Phi)_E(-1)^{-1} \\
&= (\gamma\Phi)(\lambda)\Phi(-1)(\gamma\Phi)_I(-\lambda)^{-1}(\gamma\Phi)_E(-1)^{-1} \\
&= (\gamma\Phi)_E(\lambda)\underbrace{(\gamma\Phi)_I(\lambda)\Phi(-1)(\gamma\Phi)_I(-\lambda)^{-1}(\gamma\Phi)_E(-1)^{-1}}_{\in \Lambda_I^{\varepsilon} G}.
\end{aligned}$$

Hence, by the uniqueness of the decomposition $T\big((\gamma\Phi)_E\big) = \Big(T\big((\gamma\Phi)_E\big)\Big)_E \Big(T\big((\gamma\Phi)_E\big)\Big)_I$, we have $(\gamma\Phi)_I(\lambda)\Phi(-1)(\gamma\Phi)_I(-\lambda)^{-1}(\gamma\Phi)_E(-1)^{-1} = e$ and $T\big((\gamma\Phi)_E\big) = (\gamma\Phi)_E$. ☐

Set $\mathcal{G}_0^T = \{\gamma \in \mathcal{G}_0 : \gamma(\lambda) = \gamma(-\lambda)\}$. Consider a rational loop $\gamma \in \mathcal{G}_0^T$ satisfying the reality condition $\overline{\gamma(\lambda)} = \gamma(1/\overline{\lambda})$. If γ has a singularity at $a \in \mathbb{C}^*$, then $\lambda = -a$ and $\lambda = 1/\overline{a}$ are singular points of γ. Hence there is no non-trivial simple factor in $\mathcal{G}_0^T$ (this provides a conceptual explanation to the fact that one needs two successive Bianchi-Bäcklund transforms in order to obtain a new real constant Gauss curvature surface from an old one - see [2, 14]).

Definition 3. Given a simple factor $p_{a,\xi}$, we define $\hat{p}_{a,\xi}(\lambda) = p_{a,\xi}(\lambda^2)$. Clearly, $\hat{p}_{a,\xi}$ is a rational loop satisfying the reality condition and $\hat{p}_{a,\xi} \in \mathcal{G}_0^T$. The singular dressing action $\tilde{\Phi} = \lim_{a\to 0}\left(\hat{p}_{a,\xi}\#\Phi\right)$ is a *T-simple singular dressing action.*

Let $\phi : \mathbb{C} \to N \hookrightarrow G$ be a harmonic map of finite uniton number with extended solution $\Phi : \mathbb{C} \to \Omega_T G$ and complex extended solution Ψ_μ. With the same notations used in the proof of Theorem 4, the T-simple singular dressing action of $\hat{p}_{a,\xi_{i_1}}^{-1}$ produces a new harmonic map $\hat{\phi}^{i_1}$, with complex extended solution $\hat{\Psi}^{i_1}$ and potential $\hat{\mu}^{i_1}$ given by

$$\hat{\mu}^{i_1} = \lambda^{-1} \sum_{\substack{n_I(\alpha)\geq 1\\ n_{i_1}(\alpha)=0}} B'_{1\alpha} dz + \lambda^{2n_{i_1}(\alpha)-1} \sum_{\substack{n_I(\alpha)\geq 1\\ n_{i_1}(\alpha)>0}} B'_{1\alpha} dz.$$

Again, if we continue with this procedure, we will end up with a potential $\hat{\mu}^{i_1 i_2 \dots i_k}$ with the λ^{-1} coefficient equal to zero. Hence, we have proved the following:

Theorem 6. *The harmonic map $\phi : \mathbb{C} \to N$ can be reduced to a constant by applying k T-simple singular dressing actions. Hence, any harmonic map from S^2 to N can be reduced to a constant by applying a finite number of T-simple singular dressing actions.*

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