Immersed surfaces in Lie algebras associated to primitive harmonic maps

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Abstract

Sym and Bobenko gave a construction to recover a constant mean curvature surface in 3-dimensional euclidean space from the one-parameter family of harmonic maps associated to its Gauss map into the sphere. More recently, Eschenburg and Quast generalized this construction by replacing the sphere by a Kähler symmetric space of compact type. In this paper we shall take the generalization of Eschenburg and Quast a step further: our target space is now a generalized flag manifold \( N = G/K \) and we consider immersions of \( M \) in the Lie algebra \( \mathfrak{g} \) of \( G \) associated to primitive harmonic maps.

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1 Introduction

It is well known that any non-conformal harmonic map \( \varphi \) from a simply-connected Riemann surface \( M \) into the round two-sphere \( S^2 \) is the Gauss map of a constant Gauss curvature surface, \( F : M \to \mathbb{R}^3 \), and of two parallel constant mean curvature surfaces, \( f_\pm = F \pm \varphi : M \to \mathbb{R}^3 \) (see [8] for details). Harmonic maps from a simply-connected Riemann surface into a symmetric space always come in one-parameter families and, by using a famous formulae of Sym and Bobenko [2, 11], it is possible to construct from the associated family of \( \varphi \) the three surfaces \( F, f_+, \) and \( f_- \).

Eschenburg and Quast [7] generalized this construction: they replaced the two-sphere \( S^2 \) by an arbitrary Kähler symmetric space \( N = G/K \) of compact type; they used the standard embedding to identify \( N \) with a certain adjoint orbit in the Lie algebra \( \mathfrak{g} \) of \( G \); by applying a natural generalization of Sym-Bobenko’s formulae to the associated family of an harmonic map \( \varphi : M \to N \), they obtained immersions \( F, f_+ \) and \( f_- \) of \( M \) in \( \mathfrak{g} \) and studied some of their properties. In this case, the harmonic map \( \varphi \) is not the usual Grassmannian-valued Gauss map but just a distinguished normal vector field of \( F \) and \( f_\pm \).

In the present paper we shall take the generalization of Eschenburg and Quast a step further: our target space \( N = G/K \) is now a generalized flag manifold, hence we can also
identify $N$ with a certain adjoint orbit in $\mathfrak{g}$ [3], and we consider immersions associated to primitive harmonic maps - recall that such maps also come in one-parameter families [3]. We prove that most of the properties of $f_±$ studied by Eschenburg and Quast still hold in this more general setting. Moreover, we will see that some of the geometry of the classical $S^2$ target case survives in this general setting with respect to the distinguished normal direction defined by $\varphi$: $f_+$ and $f_-$ have constant mean curvature along $\varphi$, and, if $N$ is a Kähler symmetric space and $d\varphi(TM)$ is stable under the complex structure $J$ of $N$, $F$ has constant Gauss curvature along $\varphi$.

2 Generalized Flag Manifolds

We start by recalling from [5] some facts concerning generalized flag manifolds. Let $G$ be a compact connected semisimple matrix Lie group with Lie algebra $\mathfrak{g}$. A parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ induces on $\mathfrak{g}$ a structure of graded algebra:

$$\mathfrak{g}^C = \sum_{r=-k+1}^{k-1} \mathfrak{g}_r, \quad [\mathfrak{g}_r, \mathfrak{g}_s] \subset \mathfrak{g}_{r+s},$$

(1)

with $\mathfrak{g}_- = \mathfrak{g}_{-r}$. Since $\mathfrak{g}$ is semisimple (and so it has trivial center and every derivation is an inner derivation), there exists a unique $\xi \in \mathfrak{g}$ with $\text{ad}_\mathfrak{g} \xi = i r \mathfrak{g}_r$ for all $r \in \{-k+1, \ldots, k-1\}$, the canonical element of $\mathfrak{p}$. Here and henceforth we denote $i = \sqrt{-1}$.

Let $P \subset G$ be the stabilizer of $\mathfrak{p}$ in the adjoint representation. This is a parabolic subgroup and the homogeneous space $G/\mathfrak{g}$ is a generalized flag manifold. Since $G$ is compact, $G$ acts transitively on $G/\mathfrak{g}$ so that the generalized flag manifold $G/\mathfrak{g}$ is diffeomorphic to the real coset space $G/K$, where $K = G \cap P$ has Lie algebra $\mathfrak{k} = \mathfrak{p} \cap \mathfrak{g}$.

If $W$ is a representation of $K$ we shall henceforth denote the associated bundle $G \times K W$ by $\mathcal{V}$. In particular, when $W \subset \mathfrak{g}$ is $\text{Ad}_K$-invariant, the fibre of $\mathcal{V}$ at $gK$ is given by $\mathcal{V}_{gK} = \text{Ad}_g(W)$.

Consider the inner $k$-automorphism $\tau : \mathfrak{g}^C \to \mathfrak{g}^C$ defined by

$$\tau = \text{Ad} \exp \left( \frac{2\pi i \xi}{k} \right).$$

Denote by $\omega$ the primitive $k$-th root of the unity. The $\omega^r$-eigenspace of $\tau$ is given by

$$\mathfrak{g}^r = \mathfrak{g}_r \oplus \mathfrak{g}_{r-k}.$$

In particular, $\mathfrak{g}^0 = \mathfrak{k}^C$. These eigenspaces satisfy

$$\mathfrak{g}^C = \sum_{r=0}^{k-1} \mathfrak{g}^r, \quad [\mathfrak{g}^r, \mathfrak{g}^s] \subset \mathfrak{g}^{r+s} \quad (\text{mod } k).$$

Since $\text{ad}_\xi$ takes values in $\mathfrak{g}$ when restricted to $\mathfrak{g}$, $\tau$ restricts to an automorphism of $\mathfrak{g}$, which we also denote by $\tau$. Hence we have in the generalized flag manifold $N = G/\mathfrak{g} = G/K$ a canonical $k$-symmetric structure.

Now, in this paper we shall use the $G$-equivariant map $\Xi : N \to \mathfrak{g}$ defined by $\Xi(gK) = \text{Ad}_g(\xi)$, the so called standard embedding, to identify $N$ with the adjoint orbit of $\xi$ in $\mathfrak{g}$. In particular, $\Xi$ takes values in the hypersphere of radius $(\xi, \xi)$, where $(\cdot, \cdot)$ denotes the ($G$-invariant) Killing inner product on $\mathfrak{g}$. Using $\Xi$, the tangent bundle $TN$ of $N \subset \mathfrak{g}$ is given by...
Let \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \), where
\[
\mathfrak{m}^C = \sum_{i=1}^{k-1} \mathfrak{g}^i,
\]
and the normal bundle \( TN^\perp \) is given by \( \mathfrak{t} \), that is, the orthogonal decomposition \( \mathfrak{g} = TN \oplus TN^\perp \) coincides with the homogenous reductive decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \). We shall be concerned with the metric on \( N \) induced by the Killing inner product on the ambient space \( \mathfrak{g} \) and we can define a \( G \)-invariant complex structure \( J \) on \( N \) by
\[
T^1,\mathbb{R}N = \sum_{i=1}^{k-1} [\mathfrak{g}_i].
\]

Observe that the complex structure \( J \) acting on \( [\mathfrak{g}_r] \) is just \( \frac{1}{2}\text{ad}(\Xi) \) and, when \( k = 2 \), \( N \) is a symmetric space and the metric \( \langle \cdot , \cdot \rangle \) is Kähler with respect to \( J \), that is, \( N \) is a Kähler symmetric space.

Let us compute the second fundamental form \( \alpha^N \) of \( N \rightarrow \mathfrak{g} \). First define the following endomorphism of \( TN \)
\[
I = \sum_{i=-k+1}^{-k-1} \frac{1}{r^2} \text{ad}[\Xi]_{[\mathfrak{g}_i]}.
\]
Clearly this equals \( J \) when \( k = 2 \). We have:

**Theorem 1.** Denote by \( P_{[\mathfrak{t}]} \) the orthogonal projection onto \( \mathfrak{t} \). Then
\[
\beta^N(X,Y) = P_{[\mathfrak{t}]}[IY,X],
\]
for any \( X,Y \in C^\infty(TN) \).

**Proof.** By definition, \( \alpha^N(X,Y) = P_{[\mathfrak{t}]} \partial_X Y \). Fix a point \( p \in N \) and \( u,v \in T_pN \). There exist \( \hat{X}, \hat{Y} \in [\mathfrak{m}]_p \) such that the vector fields \( X \) and \( Y \) defined by
\[
X_q = \left. \frac{d}{dt} \right|_{t=0} \exp(t\hat{X}) \cdot q = -[q,\hat{X}], \quad Y_q = \left. \frac{d}{dt} \right|_{t=0} \exp(t\hat{Y}) \cdot q = -[q,\hat{Y}],
\]
where \( \cdot \) stands for the adjoint action, satisfy \( u = X_p \) and \( v = Y_q \). Hence \( \partial_X Y = -[X,\hat{Y}] \).

Decompose \( Y \) and \( \hat{Y} \) with respect to \( (1): Y = \sum Y^r, \hat{Y} = \sum \hat{Y}^r \). Since \( Y_q = -[q,\hat{Y}] \), we have \( \hat{Y}^r = \frac{i}{2} Y^r \). Hence \( \hat{Y} = IY \), and we conclude that \( \alpha^N(u,v) = P_{[\mathfrak{t}]}[Iv,u] \). \( \square \)

### 3 Harmonic maps

Let \( G/K \) be a reductive homogeneous space, with base point \( x_0 = eK \) and reductive decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \), equipped with a left \( G \)-invariant metric. Let \( \varphi : \mathbb{C} \rightarrow G/K \) be a smooth map. Take a framing \( \psi : \mathbb{C} \rightarrow G \) of \( \varphi \), that is, we have \( \varphi = \pi \circ \psi \) where \( \pi : G \rightarrow G/K \) is the coset projection. Corresponding to the reductive decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \) there is a decomposition of \( \alpha = \psi^{-1} d\psi, \alpha = \alpha_t + \alpha_m \). It can be shown (see [4]) that \( \varphi \) is harmonic if and only if
\[
d \ast \alpha_m + [\alpha \wedge \ast \alpha_m] = 0
\]
for all local lifts \( \psi \). When \( N = G/K \) is a generalized flag manifold, we have an alternative characterization of harmonic maps:

**Theorem 2.** Let \( G/K \) be a generalized flag manifold. A smooth map \( \varphi : M \rightarrow G/K \leftarrow \mathfrak{g} \) is harmonic if and only if the 1-form \( \gamma = I \ast d\varphi \) is closed.
Proof. If \( \psi : \mathbb{C} \to G \) is a framing of \( \varphi \), we have \( \varphi = \psi \xi \psi^{-1} \), where \( \xi \) is the canonical element of \( K = P \cap G \). It is easy to check that \( \gamma = \psi \ast \alpha_m \psi^{-1} \). Hence \( d\gamma = \psi(d \ast \alpha_m + [\alpha \ast \alpha_m]) \psi^{-1} \), and we are done.

**Remark.** Consider the usual identification of \( \mathfrak{so}(3) \) with \((\mathbb{R}^3, \times)\), where \( \times \) denotes the cross product of vectors in \( \mathbb{R}^3 \). When \( N \) is the two-dimensional sphere \( S^2 = SO(3)/SO(2) \), the closeness of \( \gamma \) leads to the well-known condition of harmonicity for maps \( \varphi : \mathbb{C} \to S^2 \):

\[
d(\varphi \times *d\varphi) = 0.
\]

This means that we can integrate in order to obtain \( F : \mathbb{C} \to \mathbb{R}^3 \) with \( dF = \varphi \times *d\varphi \). Clearly, \( F \) is an immersion if and only if \( \varphi \) is an immersion. It happens that \( F \) is an immersion with constant Gauss curvature (see [8], for example). Moreover, away from umbilic points of \( F \), \( f_{\pm} = F \pm \varphi \) are immersions with constant mean curvature. When \( G/K \) is an arbitrary generalized flag manifold and \( \varphi : \mathbb{C} \to G/K \subset \mathfrak{g} \) is harmonic, we can also integrate \( \gamma \) in order to obtain \( F : \mathbb{C} \to \mathfrak{g} \) with \( dF = \gamma \) and consider \( f_{\pm} = F \pm \varphi \). Later, we shall see that, when \( \varphi \) is primitive harmonic, some of the geometry of the \( S^2 \) target case survives in this general setting with respect to a distinguished normal direction.

Recall that harmonic maps into symmetric spaces always come in one-parameter families:

If the reductive decomposition is symmetric, that is, \([\mathfrak{t}, \mathfrak{m}] \subset \mathfrak{m} \) and \([\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{t} \), then it turns out that (4) is equivalent to

\[
[\alpha_m \wedge *\alpha_m] = 0
\]

\[
d \ast \alpha_m + [\alpha \wedge *\alpha_m] = 0
\]

Now, consider the type decomposition \( \alpha_m = \alpha'_m + \alpha''_m \), where \( \alpha'_m \) is a \( \mathbb{C} \)-valued \((1,0)\)-form and \( \alpha''_m \) its complex conjugate. Consider the loop of 1-forms \( \alpha_{\lambda} = \lambda^{-1} \alpha'_m + \alpha_t + \lambda \alpha''_m \). We may view \( \alpha_{\lambda} \) as a \( \Lambda_{\mathfrak{g}} \)-valued 1-form, where

\[
\Lambda_{\mathfrak{g}} = \left\{ \xi : S^1 \to \mathfrak{g} \text{ (smooth)} \mid \tau(\xi(\lambda)) = \xi(-\lambda) \right\} \text{ for all } \lambda \in S^1 \right\}.
\]

(5)

It is easy to check that \( \varphi \) is harmonic if, and only if, \( d + \alpha_{\lambda} \) is a loop of flat connections on the trivial bundle \( \mathbb{C} \times \mathfrak{g} \). Hence, if \( \varphi \) is harmonic, we can define a smooth map \( \Psi : \mathbb{C} \to \Lambda_{\mathfrak{g}}G \), where \( \Lambda_{\mathfrak{g}}G \) is the infinite-dimensional Lie group corresponding to the loop Lie algebra (5),

\[
\Lambda_{\mathfrak{g}}G = \left\{ \gamma : S^1 \to G \text{ (smooth)} \mid \tau(\gamma(\lambda)) = \gamma(-\lambda) \right\} \text{ for all } \lambda \in S^1 \right\},
\]

such that \( \Psi^{-1} d\Psi = \alpha_{\lambda} \). The smooth map \( \Psi \) is called an extended framing (associated to \( \varphi \)). Our harmonic map is recovered from \( \Psi \) via \( \varphi = \pi \circ \Psi_1 \) (here we are using the notation \( \Psi_1(\tau) = \Psi(\tau(1)) \)).

## 4 Primitive maps

Let \( N = G/K \) be a \( k \)-symmetric space with automorphism \( \tau \) and associate eigenspace decomposition

\[
\mathfrak{g} = \bigoplus_{r=0}^{k-1} \mathfrak{g}^r.
\]

A map \( \varphi : \mathbb{C} \to G/K \) is primitive if and only if \( \alpha'_m \) takes values in \( \mathfrak{g}^1 \).
Remark. If $k > 2$ then any primitive map $\varphi : \mathbb{C} \to N$ is harmonic with respect to all invariant metrics on $N$ for which $[\mathfrak{g}_t]$ is isotropic (cf. [1]). In particular, a primitive map $\varphi : \mathbb{C} \to N$ is harmonic with respect to the metric on $N$ induced by the Killing form of $\mathfrak{g}$. Of course, when $k = 2$ the primitive condition is vacuous. Following [4], we shall talk of primitive harmonic maps whenever we want to avoid treating the case of $k$-symmetric spaces with $k = 2$ separately, although the term “primitive” (resp. “harmonic”) is redundant when $k = 2$ (resp. $k > 2$).

Primitive maps, for $k > 2$, always come in one-parameter families:

Since $\alpha_m'$ takes values in $\mathfrak{g}^1$, $\alpha_m''$ takes values in $\mathfrak{g}^{k-1}$, hence $[\alpha_m' \wedge \alpha_m''] = 0$. The projections of the Maurer-Cartan equation $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$ onto $\mathfrak{g}^1$, $\mathfrak{g}^{k-1}$ and $\mathfrak{g}^0$ are therefore given by

\begin{align}
\alpha_m' + [\alpha_t \wedge \alpha_m] &= 0 \\
\alpha_m'' + [\alpha_t \wedge \alpha_m'] &= 0 \\
d\alpha_t + \frac{1}{2}[\alpha_t \wedge \alpha_t] + [\alpha_m' \wedge \alpha_m''] &= 0
\end{align}

Consider the loop of 1-forms $\alpha_{\lambda} = \lambda^{-1} \alpha_m' + \alpha_t + \lambda \alpha_m''$. Let $\omega$ be the $k$-th primitive root of the identity. Again, we may view $\alpha_{\lambda}$ as a $\Lambda_{\tau} \mathfrak{g}$-valued 1-form, where

$$\Lambda_{\tau} \mathfrak{g} = \{ \xi : S^1 \to \mathfrak{g} \text{ (smooth)} \mid \tau(\xi(\lambda)) = \xi(\omega \lambda) \text{ for all } \lambda \in S^1 \}.$$ 

Since $\varphi$ is primitive, it is easy to check that $d + \alpha_{\lambda}$ is a loop of flat connections on the trivial bundle $\mathfrak{g}^C = \mathbb{C} \times \mathfrak{g}^C$. Hence, if $\varphi$ is harmonic, we can define a smooth map $\Psi : \mathbb{C} \to \Lambda_{\tau} G$, where $\Lambda_{\tau} G$ is the infinite-dimensional Lie group corresponding to the loop Lie algebra (5),

$$\Lambda_{\tau} G = \{ \gamma : S^1 \to G \text{ (smooth)} \mid \tau(\gamma(\lambda)) = \gamma(\omega \lambda) \text{ for all } \lambda \in S^1 \},$$

such that $\Psi^{-1} d\Psi = \alpha_{\lambda}$. Again, the smooth map $\Psi$ is called an extended framing (associated to $\varphi$). Our primitive map is recovered from $\Psi$ via $\varphi = \pi \circ \Psi_1$.

Primitive maps are well behaved with respect to homogeneous projections:

**Theorem 3.** [3] Let $K \subset H$ be closed subgroups of $G$ with $G/K$ $k$-symmetric, $k > 2$, and $M$ an almost Hermitian manifold with co-closed Kähler form. Suppose that $H$ is $\tau$-stable. If $\varphi : M \to G/K$ is a primitive map, then $p \circ \varphi : M \to G/H$ is harmonic, where $p : G/K \to G/H$ is the homogenous projection.

## 5 Immersed surfaces in the Lie algebra $\mathfrak{g}$.

Let $N = G/K$ be a generalized flag manifold with its canonical $k$-symmetric structure $\tau$, $M$ a simply-connected Riemann surface with local conformal coordinates $z = x + iy$, and $\varphi : M \to N \subset \mathfrak{g}$ a primitive immersion (not necessarily harmonic when $k=2$). Consider the following $\mathfrak{g}$-valued one-forms:

$$\theta_- = I \ast d\varphi + d\varphi, \quad \theta_+ = (k-1)I \ast d\varphi - d\varphi, \quad \theta_0 = I \ast d\varphi,$$

with $I$ given by (5). Assume that $\theta_\pm$ are injective everywhere.

**Remark.** When $k = 2$, if $\varphi$ is $J$-stable, that is, the subbundle $d\varphi(TM)$ of $\varphi^* TN$ is $J$-stable, then $\theta_{\pm}$ are both everywhere injective if and only if $\varphi$ is everywhere non-conformal.
Definition 1. An immersion \( f : M \to \mathfrak{g} \) is said an \((\pm)\)-immersion along \( \varphi \) if \( df(T^{1,0}M) = \theta_{\pm}(T^{1,0}M) \).

Given an \((\pm)\)-immersion along \( \varphi \), \( f : M \to \mathfrak{g} \), observe that its tangent bundle \( df(TM) \) is a subbundle of \( \varphi^*TN = \varphi^*[\mathfrak{m}] \). Hence, \( \varphi \), which can be viewed as a section of \( \varphi^*TN^\perp = \varphi^*[\mathfrak{t}] \), is also a section of the normal subbundle \( df(TM)^\perp \) of \( f \). We denote by \( \Pi_f \) the second fundamental form of \( f \). Consider also the second fundamental form of \( f \) with respect to \( \varphi \).

\[
\Pi_f^\varphi = \frac{1}{(\xi, \xi)}(\Pi_f, \varphi),
\]

and the first fundamental form of \( f \), \( I_f = (df, df) \). The mean curvature of \( f \) along \( \varphi \) is then given by

\[
\mathcal{H} = \frac{1}{2}\text{trace}(\Pi_f I_f^{-1}).
\]

Theorem 4. a) If \( f_\pm \) is an \((\pm)\)-immersion along \( \varphi \), then \( f_\pm \) is conformal. b) If \( f_\pm \) is an \((\pm)\)-immersion along \( \varphi \) with constant mean curvature \( \mathcal{H} \neq 0 \) along \( \varphi \), then \( \varphi \) is primitive harmonic and

\[
df_\pm = \frac{\theta_\pm}{k\mathcal{H}(\xi, \xi)}. \tag{9}
\]

c) Conversely, if \( \varphi \) is primitive harmonic, \( M \) is simply-connected, and \( \mathcal{H} \neq 0 \), there exist a pair \( f_\pm \) of \((\pm)\)-immersions along \( \varphi \) with constant mean curvature \( \mathcal{H} \) along \( \varphi \) satisfying (9).

Proof. a) Suppose that we have an \((\pm)\)-immersion \( f_\pm : M \to \mathfrak{g} \) along \( \varphi \). This means that there exists a smooth function \( a : M \to \mathbb{C} \) such that, in local coordinates,

\[
f_{\pm z} = a((i(k - 1))\varphi_z - \varphi_z). \tag{10}
\]

Write \( \varphi = \psi \xi \psi^{-1} \), with \( \psi \) a (local) framing of \( \varphi \), and \( \alpha = \psi^{-1}d\psi \). Denote by \( \alpha_m \), the \( \mathfrak{g}_z \)-component of \( \alpha_m \) and set \( A'_1 = \alpha_m(\frac{\partial}{\partial x}) \). Since \( \varphi \) is primitive, we have \( \alpha_m(\frac{\partial}{\partial z}) = A'_1 + A'_{1-k} \) and one can easily check that

\[
f_{\pm z} = kia\psi A'_1 \psi^{-1}.
\]

Since \( \mathfrak{g}_1 \) is isotropic, we conclude from here that \( f_+ \) becomes a conformal immersion.

Similarly, if \( f_- : M \to \mathfrak{g} \) along \( \varphi \) is an \((-)\)-immersion along \( \varphi \), then there exists a smooth function \( b : M \to \mathbb{C} \) such that

\[
f_{-z} = b((1)\varphi_z + \varphi_z), \tag{11}
\]

and we have

\[
f_{-z} = kib\psi A'_{1-k} \psi^{-1}.
\]

Again, since \( \mathfrak{g}_{1-k} \) is isotropic, \( f_- \) becomes a conformal immersion.

b) Let us compute the \((1, 1)\)–component of the second fundamental form of \( f_+ \). Denote by \( \mathbf{T} \) and \( \mathbf{N} \) the tangent bundle \( df_+ (TM) \) and the normal bundle \( df_+ (TM)^\perp \) of \( f_+ \), respectively. Let \( P_N : \mathfrak{g} \to \mathbf{N} \) be the orthogonal projection onto \( \mathbf{N} \). Then, in local coordinates, with \( A'' = \alpha_m(\frac{\partial}{\partial x}) \) and \( B'' = \alpha_x(\frac{\partial}{\partial y}) \), we have

\[
\begin{align*}
\Pi_{f_+}^{(1,1)} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) &= P_N f_{++z} = P_N \left\{ kia\psi A'_1 \psi^{-1} \right\} \\
&= kia P_N \left\{ \psi A'_1 \psi^{-1} + \psi[A'' + B'', A'_1] \psi^{-1} \right\} \\
&= kia P_N \left\{ A'', A'_1 \psi^{-1} + kia P_{m|m} \partial_T \psi \{ A'_1 z + [B'', A'_1] \} \psi^{-1} \right\}
\end{align*}
\]

6
In the case of equation (6) onto \( I \) along \( \phi \) by Theorem 2, \( I \) form all forms of the immersions \( b \). This gives again, by \( \phi \), since (\( \cdot,\cdot \)) is \( G \)-invariant. Then, by using the well-known identity \( (X,Y), Z = (X,[Y,Z]) \), for all \( X, Y, Z \in g \), we obtain

\[
\mathcal{H} = \frac{H^{(1,1)}_{f^+}}{H_{f^+}} = \frac{kia[A''_{k+1}, A'_{1-k}]}{k^2a^2(A'_{1-k}, A''_{k-1})} = \frac{1}{ak(\xi, \xi)}. \tag{13}
\]

Hence, since \( a \) is real, it follows from (10) and (13) that \( df^+ = \frac{\sigma_{f^+}}{\kappa H(\xi, \xi)} \). In particular, the one form \( f^+ \) is closed, that is, by Theorem 2, \( \varphi \) is harmonic.

Similarly, suppose that we have an \((-)\)-immersion \( f_- : \mathbb{C} \to g \) along \( \varphi \). In this case, the (1,1)-component of the second fundamental form of \( f_- \) is given, in local coordinates, by

\[
H^{(1,1)}_{f^-} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}} \right) = kib[A''_{k+1}, A'_{1-k}]\psi^{-1}, \tag{14}
\]

Again, \( b \) must be real and the (1,1)-component of the first fundamental form of \( f_- \) is given by

\[
I_{f^-} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}} \right) = k^2b^2(A'_{1-k}, A''_{k-1}).
\]

This gives

\[
\mathcal{H} = \frac{H^{(1,1)}_{f^-}}{I_{f^-}} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}} \right) = \frac{1}{bk(\xi, \xi)}. \tag{15}
\]

Since \( b \) is real, it follows from (11) and (15) that \( df_- = \frac{\sigma_{f_-}}{\kappa H(\xi, \xi)} \), and consequently, by Theorem 2, \( \varphi \) is harmonic.

c) Conversely, if \( M \) is simply connected, \( \mathcal{H} \neq 0 \), and \( \varphi \) is primitive harmonic, then the one-form \( f^+ d\varphi \) is closed and we can integrate in order to obtain \((\pm)\)-immersions \( f_{\pm} : M \to g \) along \( \varphi \) satisfying \( df_{\pm} = \frac{\sigma_{f_{\pm}}}{\kappa H(\xi, \xi)} \). By (10), (11), (13) and (15), we conclude that both \( f^+ \) and \( f^- \) have constant mean curvature \( \mathcal{H} \) along \( \varphi \).

\[\square\]

Remark. In the case \( k = 2 \), since \( df_{\pm} \) intertwines the complex structure \( J \) of \( N \) with the complex structure \( j \) of \( M \), the immersions \( f_{\pm} \) become \( \text{Kähler immersions} \), that is, \( j \) is an isometric parallel complex structure for the induced metric on \( M \).

The next theorem generalizes part of Theorem 7.3 in [7] and relates the second fundamental forms of the immersions \( f_{\pm} \) with that of \( N, \beta^N \):

**Theorem 5.** Let \( \varphi : M \to N \) be a primitive harmonic immersion. Consider the associated immersions \( f_{\pm} : M \to g \) satisfying (9). Then:

\[
B^{(1,1)}_{f^+} = -\mathcal{H}(\xi, \xi)\left( \beta^N(df^+, df^+) \right)^{(1,1)}, \quad B^{(1,1)}_{f^-} = \mathcal{H}(\xi, \xi)(k-1)\left( \beta^N(df^-, df^-) \right)^{(1,1)}. \tag{16}
\]
For $k > 2$, $P_{|t|}H_{f_{+}}^{(2,0)} = \left(\varphi^*\beta^N\right)^{(2,0)} = 0$, and, for $k = 2$, we have

$$P_{|t|}H_{f_{+}}^{(2,0)} = \frac{1}{2\mathcal{H}(\xi, \xi)}\left(\varphi^*\beta^N\right)^{(2,0)}, \quad P_{|t|}H_{f_{-}}^{(2,0)} = \frac{1}{2\mathcal{H}(\xi, \xi)}\left(\varphi^*\beta^N\right)^{(2,0)},$$

where $P_{|t|}$ denotes the orthogonal projection onto $[t]$.

Proof. Relations (16) follow directly from (3), (12) and (14). With respect to (17), we have:

$$P_{|t|}H_{f_{+}}^{(2,0)} \left(\frac{\partial}{\partial z} \cdot \frac{\partial}{\partial z}\right) = P_{|t|}\left\{i\frac{\hat{z}}{\mathcal{H}}\psi A_1^\prime \psi^{-1}\right\}_{z},$$

$$= \frac{i}{\mathcal{H}} P_{|t|}\left[\psi A_1^\prime \psi^{-1} + \psi[A' + B', A_1']\psi^{-1}\right] = \frac{i}{\mathcal{H}} P_{|t|}\psi[A_1' - k, A_1']\psi^{-1};$$

on the other hand, from (3) it results that

$$\beta^N(\varphi_x, \varphi_z) = \frac{k_i}{1 - k} P_{|t|}\psi[A_1' - k, A_1']\psi^{-1};$$

combine (18) with (19) and we are done. \qed

Remark. Theorems 4 and 5 admit an immediate generalization to higher dimensions by replacing the Riemann surface by a complex manifold and harmonic maps by primitive pluriharmonic maps. Recall that a smooth map from a complex manifold is primitive pluriharmonic if its restriction to any complex curve is primitive harmonic. Again, given a primitive pluriharmonic map $\varphi$ from a simply-connected complex manifold $M$ into a generalized flag manifold $G/K$, we can integrate the one-forms $\theta_z$ to obtain immersions $f_{+}$ of $M$ in $g$, which verify (16) and (17). The restrictions of $f_{+}$ and $f_{-}$ to any complex curve in $M$ have constant mean curvature along $\varphi$.

The well-known Hopf’s theorem states that any constant mean curvature immersion of the sphere $S^2$ in $\mathbb{R}^3$ is a parametrization of a round sphere. More generally:

**Theorem 6.** Suppose that $M$ is compact and $d\varphi(T^{1,1}M)$ is $I$-stable. Then the immersions $f_{+}$ and $f_{-}$ take values in hyperspheres.

Proof. Observe that $d\varphi(T^{1,1}M)$ is $I$-stable if, and only if, $\varphi_x$ takes values either in $\varphi^*[g_1]$ or $\varphi^*[g_{1-k}]$. We have

$$f_{+x} = \frac{1}{k\mathcal{H}(\xi, \xi)}(i(k-1)\varphi_x - \varphi_z).$$

If $\varphi_x$ takes values either in $\varphi^*[g_{1-k}]$, this means that $\frac{\partial f_{+x}}{\partial z} = 0$, that is, $f_{+}$ is constant on the compact $M$, by the maximum principle. If $\varphi_x$ takes values in $\varphi^*[g_1]$, we have

$$f_{+x} = - \frac{1}{\mathcal{H}(\xi, \xi)}\varphi_z.$$

Hence $\varphi + \mathcal{H}(\xi, \xi)f_{+}$ is constant. Denote this constant by $C$. Since

$$\left|f_{+} - \frac{C}{\mathcal{H}(\xi, \xi)}\right| = \frac{|\varphi|}{|\mathcal{H}(\xi, \xi)|} = \frac{1}{|\mathcal{H}|},$$

we conclude that $f_{+} : M \to g$ takes values in the hypersphere centered at $\frac{C}{\mathcal{H}(\xi, \xi)}$ and radius $\frac{1}{|\mathcal{H}|}$. Similarly, one can prove that $f_{-}$ is either constant or take values in a hypersphere. \qed
In the Kähler symmetric case \((J = J)\), \(d\varphi(T^{1,0}M)\) is \(I\)-stable if, and only if, \(\varphi\) is either holomorphic or anti-holomorphic. On the other hand, any harmonic map \(\varphi : S^2 \to S^2\) is either holomorphic or anti-holomorphic. Then, Hopf’s theorem is the particular case \(M, N = S^2\) of Theorem 6.

Now, if \(\varphi : M \to N\) is harmonic and \(M\) is simply connected, the one-form \(I + d\varphi\) is closed and we can integrate on \(M\) in order to obtain an immersion \(F : M \to g\) such that \(dF = I + d\varphi\). The Gauss curvature of \(F\) along \(\varphi\) is given by

\[
K = \det J_F^2 F^{-1}.
\]

When \(N = S^2\), it is well known that \(K\) is constant. More generally, in the Kähler symmetric case we have:

**Theorem 7.** Let \(G/K\) be a Kähler symmetric space. Suppose that \(\varphi\) is a \(J\)-stable immersion. Then the conformal structure on \(M\) is given by the second fundamental form \(F^\varphi = \frac{1}{|\xi|^2} (\nu_F, \varphi)\) and \(F\) has constant Gauss curvature \(K = \frac{1}{|\xi|^2}\) along \(\varphi\).

**Proof.** In this case, \(J = I\) and

\[
(\xi, \xi) F^\varphi = \left( \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right) (F_{zz}, \varphi) = - (F_{zz}, \varphi) = - (J \varphi_z, \varphi_z) = 0.
\]

Hence the conformal structure of \(M\) is given by \(F^\varphi\). On the other hand, since \(d\varphi(TM)\) is \(J\)-stable, \(\sin^2(\angle \varphi_z J \varphi_y) = \cos^2(\angle \varphi_z \varphi_y)\), where \(\angle XY\) denotes the angle between \(X\) and \(Y\), and we have

\[
(\xi, \xi)^2 \det F^\varphi = (F_{zz}, \varphi) (F_{yy}, \varphi) = (F_z, \varphi_z) (F_y, \varphi_y) = - (J \varphi_z, \varphi_z) (J \varphi_y, \varphi_y) = (\varphi_z, J \varphi_y)^2
\]

\[
= |\varphi_z|^2 |\varphi_y|^2 \cos^2(\angle \varphi_z J \varphi_y) = |\varphi_z|^2 |\varphi_y|^2 - |\varphi_z|^2 |\varphi_y|^2 \sin^2(\angle \varphi_z J \varphi_y)
\]

\[
= |\varphi_z|^2 |\varphi_y|^2 - (\varphi_z, \varphi_y)^2 \cos^2(\angle \varphi_z \varphi_y) = |\varphi_z|^2 |\varphi_y|^2 - (\varphi_z, \varphi_y)^2 > 0.
\]

Since

\[
\det I_F = |F_z|^2 |F_y|^2 - (F_z, F_y)^2 = |\varphi_z|^2 |\varphi_y|^2 - (\varphi_z, \varphi_y)^2,
\]

we conclude from (20) that

\[
K = \det J_F^2 I_F^{-1} = \frac{1}{(\xi, \xi)^2}.
\]

In the case \(N = S^2\), any smooth map \(\varphi : M \to S^2\) is automatically \(J\)-stable. In higher dimensions, the \(J\)-invariance is a strong restriction:

**Theorem 8.** Let \(N\) be a Kähler manifold with complex structure \(J\). A \(J\)-stable harmonic map \(\varphi : M \to N\) is either \(\pm\)-holomorphic or non-conformal.

**Proof.** Suppose that \(\varphi\) is non-constant and conformal: \((\varphi_z, \varphi_z) = 0\). Since \(\varphi\) is \(J\)-stable, we can write \(J \varphi_z = \alpha \varphi_z + \beta \varphi_{\bar{z}}\) for some smooth functions \(\alpha\) and \(\beta\). Hence

\[
0 = (J \varphi_z, \varphi_z) = \beta (\varphi_z, \varphi_{\bar{z}}).
\]

But, by the harmonicity of \(\varphi\), the singularities of \(\varphi_z\) are isolated. Hence \(\beta \equiv 0\), and consequently \(\varphi_z\) is an eigenvector of \(J\), that is, \(\varphi\) is \(\pm\)-holomorphic.

Since all harmonic maps from the sphere \(S^2\) are conformal, we see that:

**Corollary 1.** Let \(N\) be a Kähler manifold with complex structure \(J\) and \(\varphi : S^2 \to N\) a \(J\)-stable harmonic map. Then \(\varphi\) is \(\pm\)-holomorphic.
6  Sym-Bobenko’s type formulae and twistor projections

Let $N = G/K$ be a generalized flag manifold and $\varphi : M \to N$ a primitive harmonic map. Let $\Psi : M \to \Lambda rG$ be an extended framing associated to $\varphi$. We can integrate the one-forms $\theta_+$, $\theta_-$ and $\theta_0$ to obtain immersions $f_+, f_-, F : M \to \mathfrak{g}$. When $G = SO(3)$, the surfaces $f_{\pm}$ have constant mean curvature and $F$ has constant Gauss curvature, and Sym [11] and Bobenko [2] gave a formulae to recover them from the extended framing $\Psi$. More recently, Eschenburg and Quast [7] extended this construction to the Kähler symmetric space co-domain case. For primitive harmonic maps we have:

**Theorem 9.** Set $\Psi = \psi$. Then

$$F = -i \frac{\partial \Psi}{\partial \lambda} \big|_{\lambda = 1} \psi^{-1} : M \to \mathfrak{g};$$

$$f_+ = -(k - 1)i \frac{\partial \Psi}{\partial \lambda} \big|_{\lambda = 1} \psi^{-1} - \psi \xi \psi^{-1} : M \to \mathfrak{g};$$

$$f_- = -i \frac{\partial \Psi}{\partial \lambda} \big|_{\lambda = 1} \psi^{-1} + \psi \xi \psi^{-1} : M \to \mathfrak{g}.$$  

**Proof.** Since $\Psi$ is an extended framing, we have

$$\Psi^{-1} \Psi = \lambda^{-1}(A'_1 + A'_{1-k}) + B'.$$

Then, by straightforward computation, for $F$, $f_+$ and $f_-$ given by (22), one can check that

$$F_+ = i \psi(A'_1 + A'_{1-k}) \psi^{-1} = \theta_0 \left( \frac{\partial}{\partial z} \right);$$

$$f_{+} = i \psi \theta(A'_1 \psi^{-1} = \theta_+ \left( \frac{\partial}{\partial z} \right);$$

$$f_{-} = i \psi \theta(A'_{1-k} \psi^{-1} = \theta_- \left( \frac{\partial}{\partial z} \right);$$

and we are done. \qed

Consider now two generalized flag manifolds $G^C/P$ and $G^C/\bar{P}$ with their canonical $(k+1)$- and $(\bar{k}+1)$-symmetric structures $\tau$ and $\bar{\tau}$. Let $G/K$ and $G/\bar{K}$ be the corresponding real cosets. Denote by $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ the lie algebras of $P$ and $\bar{P}$, respectively. Suppose that $\mathfrak{p} \subset \bar{\mathfrak{p}}$. Then, with obvious notations, $\mathfrak{g}_j \subset \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_j$, for all $j \geq 0$ (see [10], Lemma 4.3). In particular,

$$\mathfrak{g}_0 \subset \mathfrak{g}_0, \quad \mathfrak{g}_1 \subset \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

(24)

Let $p : G/K \to G/\bar{K}$ be the homogeneous projection. From Theorem 3 and (24) it follows that if $\varphi : \mathbb{C} \to G/K$ is a primitive harmonic map then $\tilde{\varphi} = p \circ \varphi : \mathbb{C} \to G/\bar{K}$ is also primitive harmonic. Next we will see how to relate the immersed surfaces associated to $\varphi$ with those associated to $\tilde{\varphi}$.

**Lemma 1.** [10] Let $\mathfrak{g}$ be a Lie algebra, $\tau : \mathfrak{g} \to \mathfrak{g}$ an automorphism of order $k$, and $\sigma : S^1 \to \text{Aut} \, \mathfrak{g}$ a group homomorphism such that $\sigma(\omega) = \tau$, where $\omega$ is the primitive $k$-th root of the unity. Then the map $\Gamma_\tau : \Lambda \mathfrak{g} \to \Lambda \mathfrak{g}_0$ given by $\Gamma_\tau(\gamma)(\lambda) = \sigma(\lambda) \gamma(\lambda^k)$ is an isomorphism.

Starting with the canonical elements $\xi$ and $\bar{\xi}$ of $\mathfrak{p}$ and $\bar{\mathfrak{p}}$, respectively, we can define two loops of automorphisms $\sigma, \bar{\sigma} : S^1 \to \text{Aut} \, \mathfrak{g}$ by

$$\sigma(\lambda = e^{i\theta}) = \text{Ad} \exp(\theta \xi), \quad \bar{\sigma}(\lambda = e^{i\theta}) = \text{Ad} \exp(\theta \bar{\xi}).$$
Note that $\sigma(\omega) = \tau$ and $\tilde{\sigma}(\tilde{\omega}) = \tilde{\tau}$.

Then we have an isomorphism $\Gamma : \Lambda g \to \Lambda \tilde{g}$ defined by

$$\Gamma(\eta(\lambda)) = \tilde{\sigma}(\lambda^{{-k/k}})\eta(\lambda^{{k/k}}).$$

We also denote by $\Gamma : \Lambda_r G \to \Lambda_{\tilde{r}} G$ the corresponding isomorphism between loop Lie groups. On the other hand, it is easy to check that if $\Psi$ is an extended framing associated to $\varphi$, then $\tilde{\Psi} = \Gamma(\Psi)$ is an extended solution associated to $\tilde{\varphi}$. Hence to obtain the immersed surfaces $\tilde{F}$, $\tilde{f}_+$ and $\tilde{f}_-$ we only have to apply formulas (21), (22) and (23) to $\tilde{\Psi}$. Up to a translation, this gives, for example:

$$\tilde{F} = \frac{k}{k}F + \psi\left(\frac{k}{k}\xi - \xi^1\right)\psi^{-1}.$$

**Remark.** Primitive pluriharmonic maps also come in one-parameter families [6, 9]. Hence Theorem 9 still holds when we replace $M$ by a higher dimensional complex manifold and $\varphi$ by a primitive pluriharmonic map.

**References**


