Immersed surfaces in Lie algebras associated to primitive harmonic maps

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Abstract

Sym and Bobenko gave a construction to recover a constant mean curvature surface in 3-dimensional euclidean space from the one-parameter family of harmonic maps associated to its Gauss map into the sphere. More recently, Eschenburg and Quast generalized this construction by replacing the sphere by a Kähler symmetric space of compact type. In this paper we shall take the generalization of Eschenburg and Quast a step further: our target space is now a generalized flag manifold $N = G/K$ and we consider immersions of M in the Lie algebra $\mathfrak{g}$ of G associated to primitive harmonic maps.

Keywords: Primitive harmonic maps, generalized flag manifolds, immersed surfaces.
Mathematics Subject Classification 2000: 58E20, 53C43, 53C35.

1 Introduction

It is well known that any non-conformal harmonic map φ from a simply-connected Riemann surface M into the round two-sphere S^2 is the Gauss map of a constant Gauss curvature surface, $F : M \to \mathbb{R}^3$, and of two parallel constant mean curvature surfaces, $f_\pm = F \pm \varphi : M \to \mathbb{R}^3$ (see [8] for details). Harmonic maps from a simply-connected Riemann surface into a symmetric space always come in one-parameter families and, by using a famous formulae of Sym and Bobenko [2, 11], it is possible to construct from the associated family of φ the three surfaces F, f_+, and f_-.

Eschenburg and Quast [7] generalized this construction: they replaced the two-sphere S^2 by an arbitrary Kähler symmetric space $N = G/K$ of compact type; they used the standard embedding to identify N with a certain adjoint orbit in the Lie algebra $\mathfrak{g}$ of G; by applying a natural generalization of Sym-Bobenko's formulae to the associated family of an harmonic map $\varphi : M \to N$, they obtained immersions F, f_+ and f_- of M in $\mathfrak{g}$ and studied some of their properties. In this case, the harmonic map φ is not the usual Grassmannian-valued Gauss map but just a distinguished normal vector field of F and $f_\pm$.

In the present paper we shall take the generalization of Eschenburg and Quast a step further: our target space $N = G/K$ is now a generalized flag manifold, hence we can also identify N with a certain adjoint orbit in $\mathfrak{g}$ [3], and we consider immersions associated to *primitive harmonic maps* – recall that such maps also come in one-parameter families [3]. We prove that most of the properties of $f_\pm$ studied by Eschenburg and Quast still hold in this more general setting. Moreover, we will see that some of the geometry of the classical S^2 target case survives in this general setting with respect to the distinguished normal direction defined by φ: f_+ and f_- have constant mean curvature along φ, and, if N is a Kähler symmetric space and $d\varphi(TM)$ is stable under the complex structure J of N, F has constant Gauss curvature along φ.

2 Generalized Flag Manifolds

We start by recalling from [5] some facts concerning generalized flag manifolds.

Let G be a compact connected semisimple matrix Lie group with Lie algebra $\mathfrak{g}$. A parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$ induces on $\mathfrak{g}^{\mathbb{C}}$ a structure of graded algebra:

$$\mathfrak{g}^{\mathbb{C}} = \sum_{r=-k+1}^{k-1} \mathfrak{g}_r\,, \qquad [\mathfrak{g}_r, \mathfrak{g}_s] \subset \mathfrak{g}_{r+s}, \tag{1}$$

with $\overline{\mathfrak{g}_r} = \mathfrak{g}_{-r}$. Since $\mathfrak{g}$ is semisimple (and so it has trivial center and every derivation is an inner derivation), there exists a unique $\xi \in \mathfrak{g}$ with $\mathrm{ad}\xi = ir$ on $\mathfrak{g}_r$ for all $r \in \{-k+1, \dots, k-1\}$, the *canonical element* of $\mathfrak{p}$. Here and henceforth we denote $i = \sqrt{-1}$.

Let $P \subset G^{\mathbb{C}}$ be the stabilizer of $\mathfrak{p}$ in the adjoint representation. This is a *parabolic subgroup* and the homogeneous space $G^{\mathbb{C}}/P$ is a *generalized flag manifold*. Since G is compact, G acts transitively on $G^{\mathbb{C}}/P$ so that the generalized flag manifold $G^{\mathbb{C}}/P$ is diffeomorphic to the real coset space G/K, where $K = G \cap P$ has Lie algebra $\mathfrak{k} = \mathfrak{p} \cap \overline{\mathfrak{p}} \cap \mathfrak{g}$.

If W is a representation of K we shall henceforth denote the associated bundle $G \times_K W$ by $[W]$. In particular, when $W \subseteq \mathfrak{g}$ is Ad_K-invariant, the fibre of $[W]$ at gK, with $g \in G$, is given by $[W]_{gK} = \mathrm{Ad}_g(W)$.

Consider the inner k-automorphism $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ defined by

$$\tau = \mathrm{Ad}\exp\Big(\frac{2\pi\xi}{k}\Big).$$

Denote by ω the primitive k-th root of the unity. The ω^r-eigenspace of τ is given by

$$\mathfrak{g}^r = \mathfrak{g}_r \oplus \mathfrak{g}_{r-k}.$$

In particular, $\mathfrak{g}^0 = \mathfrak{k}^{\mathbb{C}}$. These eigenspaces satisfy

$$\mathfrak{g}^{\mathbb{C}} = \sum_{r=0}^{k-1} \mathfrak{g}^r\,, \qquad [\mathfrak{g}^r, \mathfrak{g}^s] \subset \mathfrak{g}^{r+s} \pmod{k}.$$

Since $\mathrm{ad}\xi$ takes values in $\mathfrak{g}$ when restricted to $\mathfrak{g}$, τ restricts to an automorphism of $\mathfrak{g}$, which we also denote by τ. Hence we have in the generalized flag manifold $N = G^{\mathbb{C}}/P = G/K$ a canonical k-symmetric structure.

Now, in this paper we shall use the G-equivariant map $\Xi : N \to \mathfrak{g}$ defined by $\Xi(gK) = \mathrm{Ad}_g(\xi)$, the so called *standard embedding*, to identify N with the adjoint orbit of ξ in $\mathfrak{g}$. In particular, Ξ takes values in the hypersphere of radius (ξ, ξ), where $(\cdot,\cdot)$ denotes the (G-invariant) Killing inner product on $\mathfrak{g}$. Using Ξ, the tangent bundle TN of $N \subset \mathfrak{g}$ is given by

$[\mathfrak{m}]$, where

$$\mathfrak{m}^{\mathbb{C}} = \sum_{r=1}^{k-1} \mathfrak{g}^r,$$

and the normal bundle $TN^{\perp}$ is given by $[\mathfrak{k}]$, that is, the orthogonal decomposition $\mathfrak{g} = TN \oplus TN^{\perp}$ coincides with the homogenous reductive decomposition $\mathfrak{g} = [\mathfrak{k}] \oplus [\mathfrak{m}]$. We shall be concerned with the metric on N induced by the Killing inner product on the ambient space $\mathfrak{g}$ and we can define a G-invariant complex structure J on N by

$$T^{1,0}N = \sum_{r=1}^{k-1} [\mathfrak{g}_r].$$

Observe that the complex structure J acting on $[\mathfrak{g}_r]$ is just $\frac{1}{r}\mathrm{ad}(\Xi)$ and, when $k = 2$, N is a symmetric space and the metric $(\cdot, \cdot)$ is Kähler with respect to J, that is, N is a Kähler symmetric space.

Let us compute the second fundamental form α^N of $N \hookrightarrow \mathfrak{g}$. First define the following endomorphism of TN

$$I = \sum_{r=-k+1}^{k-1} \frac{1}{r^2} \mathrm{ad}\Xi|_{[\mathfrak{g}_r]}. \tag{2}$$

Clearly this equals J when $k = 2$. We have:

Theorem 1. *Denote by $P_{[\mathfrak{k}]}$ the orthogonal projection onto $[\mathfrak{k}]$. Then*

$$\beta^N(X, Y) = P_{[\mathfrak{k}]}[IY, X], \tag{3}$$

for any $X, Y \in C^{\infty}(TN)$.

Proof. By definition, $\alpha^N(X, Y) = P_{[\mathfrak{k}]}\partial_X Y$. Fix a point $p \in N$ and $u, v \in T_pN$. There exist $\hat{X}, \hat{Y} \in [\mathfrak{m}]_p$ such that the vector fields X and Y defined by

$$X_q = \frac{d}{dt}\Big|_{t=0} \exp(t\hat{X}) \cdot q = -[q, \hat{X}], \quad Y_q = \frac{d}{dt}\Big|_{t=0} \exp(t\hat{Y}) \cdot q = -[q, \hat{Y}],$$

where $\cdot$ stands for the adjoint action, satisfy $u = X_p$ and $v = Y_q$. Hence $\partial_X Y = -[X, \hat{Y}]$.

Decompose Y and $\hat{Y}$ with respect to (1): $Y = \sum Y^r$, $\hat{Y} = \sum \hat{Y}^r$. Since $Y_q = -[q, \hat{Y}]$, we have $\hat{Y}^j = \frac{i}{j} Y^j$. Hence $\hat{Y} = IY$, and we conclude that $\alpha^N(u, v) = P_{[\mathfrak{k}]_p}[Iv, u]$. $\square$

3 Harmonic maps

Let G/K be a reductive homogeneous space, with base point $x_0 = eK$ and reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, equipped with a left G-invariant metric. Let $\varphi : \mathbb{C} \to G/K$ be a smooth map. Take a framing $\psi : \mathbb{C} \to G$ of φ, that is, we have $\varphi = \pi \circ \psi$ where $\pi : G \to G/K$ is the coset projection. Corresponding to the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ there is a decomposition of $\alpha = \psi^{-1}d\psi$, $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$. It can be shown (see [4]) that φ is harmonic if and only if

$$d * \alpha_{\mathfrak{m}} + [\alpha \wedge *\alpha_{\mathfrak{m}}] \quad = \quad 0 \tag{4}$$

for all local lifts ψ. When $N = G/K$ is a generalized flag manifold, we have an alternative characterization of harmonic maps:

Theorem 2. *Let G/K be a generalized flag manifold. A smooth map $\varphi : M \to G/K \hookrightarrow \mathfrak{g}$ is harmonic if and only if the 1-form $\gamma = I * d\varphi$ is closed.*

Proof. If $\psi : \mathbb{C} \to G$ is a framing of φ, we have $\varphi = \psi\xi\psi^{-1}$, where ξ is the canonical element of $K = P \cap G$. It is easy to check that $\gamma = \psi * \alpha_{\mathfrak{m}}\psi^{-1}$. Hence $d\gamma = \psi\{d * \alpha_{\mathfrak{m}} + [\alpha \wedge *\alpha_{\mathfrak{m}}]\}\psi^{-1}$, and we are done. □

Remark. Consider the usual identification of $\mathfrak{so}(3)$ with $(\mathbb{R}^3, \times)$, where $\times$ denotes the cross product of vectors in $\mathbb{R}^3$. When N is the two-dimensional sphere $S^2 = SO(3)/SO(2)$, the closeness of γ leads to the well-known condition of harmonicity for maps $\varphi : \mathbb{C} \to S^2$:

$$d(\varphi \times *d\varphi) = 0.$$

This means that we can integrate in order to obtain $F : \mathbb{C} \to \mathbb{R}^3$ with $dF = \varphi \times *d\varphi$. Clearly, F is an immersion if and only if φ is an immersion. It happens that F is an immersion with constant Gauss curvature (see [8], for example). Moreover, away from umbilic points of F, $f_{\pm} = F \pm \varphi$ are immersions with constant mean curvature. When G/K is an arbitrary generalized flag manifold and $\varphi : \mathbb{C} \to G/K \subset \mathfrak{g}$ is harmonic, we can also integrate γ in order to obtain $F : \mathbb{C} \to \mathfrak{g}$ with $dF = \gamma$ and consider $f_{\pm} = F \pm \varphi$. Later, we shall see that, when φ is *primitive harmonic*, some of the geometry of the S^2 target case survives in this general setting with respect to a distinguished normal direction.

Recall that harmonic maps into symmetric spaces always come in one-parameter families: If the reductive decomposition is symmetric, that is, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, then it turns out that (4) is equivalent to

$$\begin{aligned} [\alpha_{\mathfrak{m}} \wedge *\alpha_{\mathfrak{m}}] &= 0 \\ d * \alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge *\alpha_{\mathfrak{m}}] &= 0 \end{aligned}$$

Now, consider the type decomposition $\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$, where $\alpha'_{\mathfrak{m}}$ is a $\mathfrak{m}^{\mathbb{C}}$-valued $(1,0)$-form and $\alpha''_{\mathfrak{m}}$ its complex conjugate. Consider the loop of 1-forms $\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}}$. We may view α_λ as a $\Lambda_\tau\mathfrak{g}$-valued 1-form, where

$$\Lambda_\tau\mathfrak{g} = \left\{\xi : S^1 \to \mathfrak{g} \text{ (smooth)} \mid \tau(\xi(\lambda)) = \xi(-\lambda) \text{ for all } \lambda \in S^1\right\}. \tag{5}$$

It is easy to check that φ is harmonic if, and only if, $d + \alpha_\lambda$ is a loop of flat connections on the trivial bundle $\underline{\mathfrak{g}}^{\mathbb{C}} = \mathbb{C} \times \mathfrak{g}^{\mathbb{C}}$. Hence, if φ is harmonic, we can define a smooth map $\Psi : \mathbb{C} \to \Lambda_\tau G$, where $\Lambda_\tau G$ is the infinite-dimensional Lie group corresponding to the loop Lie algebra (5),

$$\Lambda_\tau G = \left\{\gamma : S^1 \to G \text{ (smooth)} \mid \tau(\gamma(\lambda)) = \gamma(-\lambda) \text{ for all } \lambda \in S^1\right\},$$

such that $\Psi^{-1}d\Psi = \alpha_\lambda$. The smooth map Ψ is called an *extended framing* (associated to φ). Our harmonic map is recovered from Ψ via $\varphi = \pi \circ \Psi_1$ (here we are using the notation $\Psi_1(z) = \Psi(z)(1)$).

4 Primitive maps

Let $N = G/K$ be a k-symmetric space with automorphism τ and associate eigenspace decomposition

$$\mathfrak{g}^{\mathbb{C}} = \sum_{r=0}^{k-1} \mathfrak{g}^r.$$

A map $\varphi : \mathbb{C} \to G/K$ is *primitive* if and only if $\alpha'_{\mathfrak{m}}$ takes values in $\mathfrak{g}^1$.

Remark. If $k > 2$ then any primitive map $\varphi : \mathbb{C} \to N$ is harmonic with respect to all invariant metrics on N for which $[\mathfrak{g}_1]$ is isotropic (cf. [1]). In particular, a primitive map $\varphi : \mathbb{C} \to N$ is harmonic with respect to the metric on N induced by the Killing form of $\mathfrak{g}$. Of course, when $k = 2$ the primitive condition is vacuous. Following [4], we shall talk of *primitive harmonic maps* whenever we want to avoid treating the case of k-symmetric spaces with $k = 2$ separately, although the term "primitive" (resp. "harmonic") is redundant when $k = 2$ (resp. $k > 2$).

Primitive maps, for $k > 2$, always come in one-parameter families:

Since $\alpha'_{\mathfrak{m}}$ takes values in $\mathfrak{g}^1$, $\alpha''_{\mathfrak{m}}$ takes values in $\overline{\mathfrak{g}^1} = \mathfrak{g}^{k-1}$, hence $[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}} = 0$. The projections of the Maurer-Cartan equation $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$ onto $\mathfrak{g}^1$, $\mathfrak{g}^{k-1}$ and $\mathfrak{g}^0$ are therefore given by

$$d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}] = 0 \tag{6}$$

$$d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}] = 0 \tag{7}$$

$$d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}] = 0 \tag{8}$$

Consider the loop of 1-forms $\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}}$. Let ω be the k-th primitive root of the identity. Again, we may view α_λ as a $\Lambda_\tau\mathfrak{g}$-valued 1-form, where

$$\Lambda_\tau\mathfrak{g} = \left\{\xi : S^1 \to \mathfrak{g} \text{ (smooth)} \mid \tau(\xi(\lambda)) = \xi(\omega\lambda) \text{ for all } \lambda \in S^1\right\}.$$

Since φ is primitive, it is easy to check that $d + \alpha_\lambda$ is a loop of flat connections on the trivial bundle $\underline{\mathfrak{g}}^{\mathbb{C}} = \mathbb{C} \times \mathfrak{g}^{\mathbb{C}}$. Hence, if φ is harmonic, we can define a smooth map $\Psi : \mathbb{C} \to \Lambda_\tau G$, where $\Lambda_\tau G$ is the infinite-dimensional Lie group corresponding to the loop Lie algebra (5),

$$\Lambda_\tau G = \left\{\gamma : S^1 \to G \text{ (smooth)} \mid \tau(\gamma(\lambda)) = \gamma(\omega\lambda) \text{ for all } \lambda \in S^1\right\},$$

such that $\Psi^{-1}d\Psi = \alpha_\lambda$. Again, the smooth map Ψ is called an *extended framing* (associated to φ). Our primitive map is recovered from Ψ via $\varphi = \pi \circ \Psi_1$.

Primitive maps are well behaved with respect to homogeneous projections:

Theorem 3. [3] *Let $K \subset H$ be closed subgroups of G with G/K k-symmetric, $k > 2$, and M an almost Hermitian manifold with co-closed Kähler form. Suppose that H is τ-stable. If $\varphi : M \to G/K$ is a primitive map, then $p \circ \varphi : M \to G/H$ is harmonic, where $p : G/K \to G/H$ is the homogenous projection.*

5 Immersed surfaces in the Lie algebra $\mathfrak{g}$.

Let $N = G/K$ be a generalized flag manifold with its canonical k-symmetric structure τ, M a simply-connected Riemann surface with local conformal coordinates $z = x + iy$, and $\varphi : M \to N \subset \mathfrak{g}$ a primitive immersion (not necessarily harmonic when k=2). Consider the following $\mathfrak{g}$-valued one-forms:

$$\theta_- = I * d\varphi + d\varphi, \quad \theta_+ = (k-1)I * d\varphi - d\varphi, \quad \theta_0 = I * d\varphi,$$

with I given by (5). Assume that $\theta_\pm$ are injective everywhere.

Remark. When $k = 2$, if φ is *J-stable*, that is, the subbundle $d\varphi(TM)$ of φ^*TN is J-stable, then $\theta_\pm$ are both everywhere injective if and only if φ is everywhere non-conformal.

Definition 1. An immersion $f : M \to \mathfrak{g}$ is said an $(\pm)$-*immersion along* φ if $df(T^{(1,0)}M) = \theta_\pm(T^{(1,0)}M)$.

Given an $(\pm)$-immersion along φ, $f : M \to \mathfrak{g}$, observe that its tangent bundle $df(TM)$ is a subbundle of $\varphi^*TN = \varphi^*[\mathfrak{m}]$. Hence, φ, which can be viewed as a section of $\varphi^*TN^\perp = \varphi^*[\mathfrak{k}]$, is also a section of the normal subbundle $df(TM)^\perp$ of f. We denote by $I\!I_f$ the second fundamental form of f. Consider also the second fundamental form of f with respect to φ,

$$I\!I_f^\varphi = \frac{1}{(\xi,\xi)}(I\!I_f, \varphi),$$

and the first fundamental form of f, $I_f = (df, df)$. The *mean curvature of* f *along* φ is then given by

$$\mathcal{H} = \frac{1}{2}\mathrm{trace}(I\!I_f^\varphi I_f^{-1}).$$

Theorem 4. *a) If* $f_\pm$ *is an* $(\pm)$*-immersion along* φ*, then* $f_\pm$ *is conformal. b) If* $f_\pm$ *is an* $(\pm)$*-immersion along* φ *with constant mean curvature* $\mathcal{H} \neq 0$ *along* φ*, then* φ *is primitive harmonic and*

$$df_\pm = \frac{\theta_\pm}{k\mathcal{H}(\xi,\xi)}. \tag{9}$$

c) Conversely, if φ *is primitive harmonic,* M *is simply-connected, and* $\mathcal{H} \neq 0$*, there exist a pair* $f_\pm$ *of* $(\pm)$*-immersions along* φ *with constant mean curvature* $\mathcal{H}$ *along* φ *satisfying (9).*

Proof. a) Suppose that we have an $(+)$-immersion $f_+ : M \to \mathfrak{g}$ along φ. This means that there exists a smooth function $a : M \to \mathbb{C}$ such that, in local coordinates,

$$f_{+z} = a\big(i(k-1)I\varphi_z - \varphi_z\big). \tag{10}$$

Write $\varphi = \psi\xi\psi^{-1}$, with ψ a (local) framing of φ, and $\alpha = \psi^{-1}d\psi$. Denote by $\alpha_{\mathfrak{m}\,r}$ the $\mathfrak{g}_r$-component of $\alpha_\mathfrak{m}$ and set $A'_r = \alpha_{\mathfrak{m}\,r}\big(\frac{\partial}{\partial z}\big)$. Since φ is primitive, we have $\alpha_\mathfrak{m}\big(\frac{\partial}{\partial z}\big) = A'_1 + A'_{1-k}$ and one can easily check that

$$f_{+z} = kia\psi A'_1\psi^{-1}.$$

Since $\mathfrak{g}_1$ is isotropic, we conclude from here that f_+ becomes a conformal immersion.

Similarly, if $f_- : M \to \mathfrak{g}$ along φ is an $(-)$-immersion along φ, then there exists a smooth function $b : M \to \mathbb{C}$ such that

$$f_{-z} = b\big(iI\varphi_z + \varphi_z\big), \tag{11}$$

and we have

$$f_{-z} = kib\psi A'_{1-k}\psi^{-1}.$$

Again, since $\mathfrak{g}_{1-k}$ is isotropic, f_- becomes a conformal immersion.

b) Let us compute the $(1,1)$-component of the second fundamental form of f_+. Denote by $\mathbf{T}$ and $\mathbf{N}$ the tangent bundle $df_+(TM)$ and the normal bundle $df_+(TM)^\perp$ of f_+, respectively. Let $P_\mathbf{N} : \mathfrak{g} \to \mathbf{N}$ be the orthogonal projection onto $\mathbf{N}$. Then, in local coordinates, with $A'' = \alpha_\mathfrak{m}\big(\frac{\partial}{\partial \bar z}\big)$ and $B'' = \alpha_\mathfrak{k}\big(\frac{\partial}{\partial \bar z}\big)$, we have

$$\begin{aligned} I\!I^{(1,1)}_{f_+}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}\Big) &= P_\mathbf{N} f_{+\bar z z} = P_\mathbf{N}\big\{kia\psi A'_1\psi^{-1}\big\}_{\bar z} \\ &= kiaP_\mathbf{N}\big\{\psi A'_{1\bar z}\psi^{-1} + \psi[A'' + B'', A'_1]\psi^{-1}\big\} \\ &= kia\psi[A'', A'_1]\psi^{-1} + kiaP_{\varphi^*[\mathfrak{m}]\ominus\mathbf{T}}\psi\big\{A'_{1\bar z} + [B'', A'_1]\big\}\psi^{-1} \end{aligned}$$

Where $\varphi^*[\mathfrak{m}] \ominus \mathbf{T}$ denotes the orthogonal complement of $\mathbf{T}$ in $\varphi^*[\mathfrak{m}]$. Observe that the projection of equation (6) onto $\mathfrak{g}_1$ gives $A'_{1\bar{z}} + [B'', A'_1] = 0$. On the other hand, A'' takes values in $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{k-1}$. Hence:

$$II^{(1,1)}_{f_+}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\Big) = kia\psi[A''_{-1}, A'_1]\psi^{-1}. \tag{12}$$

Clearly, since $II^{(1,1)}_{f_+}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\Big)$ is real and $[A''_{-1}, A'_1]$ is imaginary, a must be real.

Now, the $(1,1)$-component of the first fundamental form of f_+ is given by

$$I_{f_+}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\Big) = k^2a^2\big(\psi A'_1\psi^{-1}, \psi A''_{-1}\psi^{-1}\big) = k^2a^2\big(A'_1, A''_{-1}\big),$$

since $(\cdot,\cdot)$ is G-invariant. Then, by using the well-known identity $([X,Y],Z) = (X,[Y,Z])$, for all $X,Y,Z \in \mathfrak{g}$, we obtain

$$\mathcal{H} = \frac{II^{\varphi\ (1,1)}_{f+}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\Big)}{I_{f_+}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\Big)} = \frac{kia([A''_{-1}, A'_1], \xi)}{k^2a^2\big(A'_1, A''_{-1}\big)(\xi,\xi)} = \frac{1}{ak(\xi,\xi)}. \tag{13}$$

Hence, since a is real, it follows from (10) and (13) that $df_+ = \frac{\theta_+}{k\mathcal{H}(\xi,\xi)}$. In particular, the one form $I * d\varphi$, being exact (in fact, $F = \frac{1}{k-1}(k\mathcal{H}(\xi,\xi)f_+ + \varphi)$ integrates it), is closed, that is, by Theorem 2, φ is harmonic.

Similarly, suppose that we have an $(-)$-immersion $f_- : \mathbb{C} \to \mathfrak{g}$ along φ. In this case, the $(1,1)$-component of the second fundamental form of f_- is given, in local coordinates, by

$$II^{(1,1)}_{f_-}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\Big) = kib\psi[A''_{k-1}, A'_{1-k}]\psi^{-1}, \tag{14}$$

Again, b must be real and the $(1,1)$-component of the first fundamental form of f_- is given by

$$I_{f_-}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\Big) = k^2b^2\big(A'_{1-k}, A''_{k-1}\big),$$

This gives

$$\mathcal{H} = \frac{II^{\varphi\ (1,1)}_{f_-}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\Big)}{I_{f_-}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\Big)} = \frac{1}{bk(\xi,\xi)}. \tag{15}$$

Since b is real, it follows from (11) and (15) that $df_- = \frac{\theta_-}{k\mathcal{H}(\xi,\xi)}$, and consequently, by Theorem 2, φ is harmonic.

c) Conversely, if M is simply connected, $\mathcal{H} \neq 0$, and φ is primitive harmonic, then the one-form $I * d\varphi$ is closed and we can integrate in order to obtain $(\pm)$-immersions $f_\pm : M \to \mathfrak{g}$ along φ satisfying $df_\pm = \frac{\theta_\pm}{k\mathcal{H}(\xi,\xi)}$. By (10), (11), (13) and (15), we conclude that both f_+ and f_- have constant mean curvature $\mathcal{H}$ along φ. □

Remark. In the case $k = 2$, since $df_\pm$ intertwines the complex structure J of N with the complex structure j of M, the immersions $f_\pm$ become *Kähler immersions*, that is, j is an isometric parallel complex structure for the induced metric on M.

The next theorem generalizes part of Theorem 7.3 in [7] and relates the second fundamental forms of the immersions $f_\pm$ with that of N, β^N:

Theorem 5. *Let $\varphi : M \to N$ be a primitive harmonic immersion. Consider the associated immersions $f_\pm : M \to \mathfrak{g}$ satisfying (9). Then:*

$$II^{(1,1)}_{f_+} = -\mathcal{H}(\xi,\xi)\Big(\beta^N\big(df_+, df_+\big)\Big)^{(1,1)},\ \ II^{(1,1)}_{f_-} = \mathcal{H}(\xi,\xi)(k-1)\Big(\beta^N\big(df_-, df_-\big)\Big)^{(1,1)}. \tag{16}$$

For $k>2$, $P_{[\mathfrak{k}]}II^{(2,0)}_{f\pm}=\Big(\varphi^*\beta^N\Big)^{(2,0)}=0$, *and, for* $k=2$, *we have*

$$P_{[\mathfrak{k}]}II^{(2,0)}_{f_+}=-\frac{1}{2\mathcal{H}(\xi,\xi)}\Big(\varphi^*\beta^N\Big)^{(2,0)},\ \ P_{[\mathfrak{k}]}II^{(2,0)}_{f_-}=\frac{1}{2\mathcal{H}(\xi,\xi)}\Big(\varphi^*\beta^N\Big)^{(2,0)}, \tag{17}$$

where $P_{[\mathfrak{k}]}$ *denotes the orthogonal projection onto* $[\mathfrak{k}]$.

Proof. Relations (16) follow directly from (3), (12) and (14). With respect to (17), we have:

$$\begin{aligned}P_{[\mathfrak{k}]}II^{(2,0)}_{f_+}\Big(\frac{\partial}{\partial z},\frac{\partial}{\partial z}\Big)&=P_{[\mathfrak{k}]}\Big\{\frac{i}{(\xi,\xi)\mathcal{H}}\psi A'_1\psi^{-1}\Big\}_z\\&=\frac{i}{(\xi,\xi)\mathcal{H}}P_{[\mathfrak{k}]}\{\psi A'_{1z}\psi^{-1}+\psi[A'+B',A'_1]\psi^{-1}\}\\&=\frac{i}{(\xi,\xi)\mathcal{H}}P_{[\mathfrak{k}]}\psi\big[A'_{1-k},A'_1\big]\psi^{-1};\end{aligned}\tag{18}$$

on the other hand, from (3) it results that

$$\beta^N(\varphi_z,\varphi_z)=\frac{ki}{1-k}P_{[\mathfrak{k}]}\psi[A'_{1-k},A'_1]\psi^{-1};\tag{19}$$

combine (18) with (19) and we are done. □

Remark. Theorems 4 and 5 admit an immediate generalization to higher dimensions by replacing the Riemann surface by a complex manifold and harmonic maps by *primitive pluriharmonic maps*. Recall that a smooth map from a complex manifold is primitive pluriharmonic if its restriction to any complex curve is primitive harmonic. Again, given a primitive pluriharmonic map φ from a simply-connected complex manifold M into a generalized flag manifold G/K, we can integrate the one-forms $\theta_\pm$ to obtain immersions $f_\pm$ of M in $\mathfrak{g}$, which verify (16) and (17). The restrictions of f_+ and f_- to any complex curve in M have constant mean curvature along φ.

The well-known Hopf's theorem states that any constant mean curvature immersion of the sphere S^2 in $\mathbb{R}^3$ is a parametrization of a round sphere. More generally:

Theorem 6. *Suppose that* M *is compact and* $d\varphi(T^{(1,0)}M)$ *is* I*-stable. Then the immersions* f_+ *and* f_- *take values in hyperspheres.*

Proof. Observe that $d\varphi(T^{(1,0)}M)$ is I-stable if, and only if, φ_z takes values either in $\varphi^*[\mathfrak{g}_1]$ or $\varphi^*[\mathfrak{g}_{1-k}]$. We have

$$f_{+z}=\frac{1}{k\mathcal{H}(\xi,\xi)}\big(i(k-1)I\varphi_z-\varphi_z\big).$$

If φ_z takes values either in $\varphi^*[\mathfrak{g}_{1-k}]$, this means that $\frac{\partial f_+}{\partial z}=0$, that is, f_+ is constant on the compact M, by the maximum principle. If φ_z takes values in $\varphi^*[\mathfrak{g}_1]$, we have

$$f_{+z}=-\frac{1}{\mathcal{H}(\xi,\xi)}\varphi_z.$$

Hence $\varphi+\mathcal{H}(\xi,\xi)f_+$ is constant. Denote this constant by C. Since

$$\Big|f_+-\frac{C}{\mathcal{H}(\xi,\xi)}\Big|=\frac{|\varphi|}{|\mathcal{H}|(\xi,\xi)}=\frac{1}{|\mathcal{H}|},$$

we conclude that $f_+:M\to\mathfrak{g}$ takes values in the hypersphere centered at $\frac{C}{\mathcal{H}(\xi,\xi)}$ and radius $\frac{1}{|\mathcal{H}|}$. Similarly, one can prove that f_- is either constant or take values in a hypersphere. □

In the Kähler symmetric case ($I = J$), $d\varphi(T^{(1,0)}M)$ is I-stable if, and only if, φ is either holomorphic or anti-holomorphic. On the other hand, any harmonic map $\varphi : S^2 \to S^2$ is either holomorphic or anti-holomorphic. Then, Hopf's theorem is the particular case $M, N = S^2$ of Theorem 6.

Now, if $\varphi : M \to N$ is harmonic and M is simply connected, the one-form $I * d\varphi$ is closed and we can integrate on M in order to obtain an immersion $F : M \to \mathfrak{g}$ such that $dF = I * d\varphi$. The *Gauss curvature of F along φ* is given by

$$\mathcal{K} = \det II_F^{\varphi} I_F^{-1}.$$

When $N = S^2$, it is well known that $\mathcal{K}$ is constant. More generally, in the Kähler symmetric case we have:

Theorem 7. *Let G/K be a Kähler symmetric space. Suppose that φ is a J-stable immersion. Then the conformal structure on M is given by the second fundamental form $II_F^{\varphi} = \frac{1}{(\xi,\xi)}(II_F, \varphi)$ and F has constant Gauss curvature $\mathcal{K} = \frac{1}{(\xi,\xi)^2}$ along φ.*

Proof. In this case, $J = I$ and

$$(\xi,\xi) II_F^{\varphi}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\Big) = \big(F_{zz}, \varphi\big) = -\big(F_z, \varphi_z\big) = -i\big(J\varphi_z, \varphi_z\big) = 0.$$

Hence the conformal structure of M is given by II_F^{φ}. On the other hand, since $d\varphi(TM)$ is J-stable, $\sin^2\big(\angle\varphi_x J\varphi_y\big) = \cos^2\big(\angle\varphi_x\varphi_y\big)$, where $\angle XY$ denotes the angle between X and Y, and we have

$$\begin{aligned}(\xi,\xi)^2 \det II_F^{\varphi} &= \big(F_{xx}, \varphi\big)\big(F_{yy}, \varphi\big) = \big(F_x, \varphi_x\big)\big(F_y, \varphi_y\big) = -\big(J\varphi_y, \varphi_x\big)\big(J\varphi_x, \varphi_y\big) = \big(\varphi_x, J\varphi_y\big)^2 \\ &= |\varphi_x|^2|\varphi_y|^2 \cos^2\big(\angle\varphi_x J\varphi_y\big) = |\varphi_x|^2|\varphi_y|^2 - |\varphi_x|^2|\varphi_y|^2 \sin^2\big(\angle\varphi_x J\varphi_y\big) \\ &= |\varphi_x|^2|\varphi_y|^2 - |\varphi_x|^2|\varphi_y|^2 \cos^2\big(\angle\varphi_x\varphi_y\big) = |\varphi_x|^2|\varphi_y|^2 - \big(\varphi_x, \varphi_y\big)^2 > 0. \qquad (20)\end{aligned}$$

Since

$$\det I_F = |F_x|^2|F_y|^2 - \big(F_x, F_y\big)^2 = |\varphi_x|^2|\varphi_y|^2 - \big(\varphi_x, \varphi_y\big)^2,$$

we conclude from (20) that

$$\mathcal{K} = \det II_F^{\varphi} I_F^{-1} = \frac{1}{(\xi,\xi)^2}.$$

□

In the case $N = S^2$, any smooth map $\varphi : M \to S^2$ is automatically J-stable. In higher dimensions, the J-invariance is a strong restriction:

Theorem 8. *Let N be a Kähler manifold with complex structure J. A J-stable harmonic map $\varphi : M \to N$ is either $\pm$-holomorphic or non-conformal.*

Proof. Suppose that φ is non-constant and conformal: $(\varphi_z, \varphi_z) = 0$. Since φ is J-stable, we can write $J\varphi_z = \alpha\varphi_z + \beta\varphi_{\bar z}$ for some smooth functions α and β. Hence

$$0 = (J\varphi_z, \varphi_z) = \beta(\varphi_z, \varphi_{\bar z}).$$

But, by the harmonicity of φ, the singularities of φ_z are isolated. Hence $\beta \equiv 0$, and consequently φ_z is an eigenvector of J, that is, φ is $\pm$-holomorphic. □

Since all harmonic maps from the sphere S^2 are conformal, we see that:

Corollary 1. *Let N be a Kähler manifold with complex structure J and $\varphi : S^2 \to N$ a J-stable harmonic map. Then φ is $\pm$-holomorphic.*

6 Sym-Bobenko's type formulae and twistor projections

Let $N = G/K$ be a generalized flag manifold and $\varphi : M \to N$ a primitive harmonic map. Let $\Psi : M \to \Lambda_\tau G$ be an extended framing associated to φ. We can integrate the one-forms θ_+, θ_- and θ_0 to obtain immersions $f_+, f_-, F : M \to \mathfrak{g}$. When $G = SO(3)$, the surfaces $f_\pm$ have constant mean curvature and F has constant Gauss curvature, and Sym [11] and Bobenko [2] gave a formulae to recover them from the extended framing Ψ. More recently, Eschenburg and Quast [7] extended this construction to the Kähler symmetric space co-domain case. For primitive harmonic maps we have:

Theorem 9. *Set $\Psi_1 = \psi$. Then*

$$F = -i\,\frac{\partial \Psi}{\partial \lambda}\Big|_{\lambda=1}\psi^{-1} : M \to \mathfrak{g}; \tag{21}$$

$$f_+ = -(k-1)i\,\frac{\partial \Psi}{\partial \lambda}\Big|_{\lambda=1}\psi^{-1} - \psi\xi\psi^{-1} : M \to \mathfrak{g}; \tag{22}$$

$$f_- = -i\,\frac{\partial \Psi}{\partial \lambda}\Big|_{\lambda=1}\psi^{-1} + \psi\xi\psi^{-1} : M \to \mathfrak{g}. \tag{23}$$

Proof. Since Ψ is an extended framing, we have

$$\Psi^{-1}\Psi_z = \lambda^{-1}(A_1' + A_{1-k}') + B'.$$

Then, by straightforward computation, for F, f_+ and f_- given by (22), one can check that

$$F_z = i\psi(A_1' + A_{1-k}')\psi^{-1} = \theta_0\Big(\frac{\partial}{\partial z}\Big);$$
$$f_{+z} = ik\psi A_1'\psi^{-1} = \theta_+\Big(\frac{\partial}{\partial z}\Big);$$
$$f_{-z} = ik\psi A_{1-k}'\psi^{-1} = \theta_-\Big(\frac{\partial}{\partial z}\Big);$$

and we are done. □

Consider now two generalized flag manifolds $G^{\mathbb{C}}/P$ and $G^{\mathbb{C}}/\tilde{P}$ with their canonical $(k+1)$- and $(\tilde{k}+1)$-symmetric structures τ and $\tilde{\tau}$. Let G/K and $G/\tilde{K}$ be the corresponding real cosets. Denote by $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$ the lie algebras of P and $\tilde{P}$, respectively. Suppose that $\mathfrak{p} \subset \tilde{\mathfrak{p}}$. Then, with obvious notations, $\mathfrak{g}_j \subset \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \ldots \oplus \tilde{\mathfrak{g}}_j$, for all $j \geq 0$ (see [10], Lemma 4.3). In particular,

$$\mathfrak{g}_0 \subset \tilde{\mathfrak{g}}_0, \text{ and } \mathfrak{g}_1 \subset \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \tag{24}$$

Let $p : G/K \to G/\tilde{K}$ be the homogeneous projection. From Theorem 3 and (24) it follows that if $\varphi : \mathbb{C} \to G/K$ is a primitive harmonic map then $\tilde{\varphi} = p \circ \varphi : \mathbb{C} \to G/\tilde{K}$ is also primitive harmonic. Next we will see how to relate the immersed surfaces associated to φ with those associated to $\tilde{\varphi}$.

Lemma 1. [10] *Let $\mathfrak{g}$ be a Lie algebra, $\tau : \mathfrak{g} \to \mathfrak{g}$ an automorphism of order k, and $\sigma : S^1 \to$* Aut $\mathfrak{g}$ *a group homomorphism such that $\sigma(\omega) = \tau$, where ω is the primitive k-th root of the unity. Then the map $\Gamma_\tau : \Lambda\mathfrak{g} \to \Lambda\mathfrak{g}_\tau$ given by $\Gamma_\tau(\gamma)(\lambda) = \sigma(\lambda)\gamma(\lambda^k)$ is an isomorphism.*

Starting with the canonical elements ξ and $\tilde{\xi}$ of $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$, respectively, we can define two loops of automorphisms $\sigma, \tilde{\sigma} : S^1 \to \mathrm{Aut}\,\mathfrak{g}$ by

$$\sigma(\lambda = e^{i\theta}) = \mathrm{Ad}\exp(\theta\xi) \qquad \tilde{\sigma}(\lambda = e^{i\theta}) = \mathrm{Ad}\exp(\theta\tilde{\xi}).$$

Note that $\sigma(\omega) = \tau$ and $\tilde{\sigma}(\tilde{\omega}) = \tilde{\tau}$.

Then we have an isomorphism $\Gamma : \Lambda\mathfrak{g}_\tau \to \Lambda\mathfrak{g}_{\tilde{\tau}}$ defined by

$$\Gamma(\eta)(\lambda) = \tilde{\sigma}(\lambda)\sigma(\lambda^{-\tilde{k}/k})\eta(\lambda^{k/\tilde{k}}).$$

We also denote by $\Gamma : \Lambda_\tau G \to \Lambda_{\tilde{\tau}} G$ the corresponding isomorphism between loop Lie groups. On the other hand, it is easy to check that if Ψ is an extended framing associated to φ, then $\tilde{\Psi} = \Gamma(\Psi)$ is an extended solution associated to $\tilde{\varphi}$. Hence to obtain the immersed surfaces $\tilde{F}$, $\tilde{f}_+$ and $\tilde{f}_-$ we only have to apply formulas (21), (22) and (23) to $\tilde{\Psi}$. Up to a translation, this gives, for example:

$$\tilde{F} = \frac{\tilde{k}}{k} F + \psi\Big(\frac{\tilde{k}}{k}\xi - \tilde{\xi}\Big)\psi^{-1}.$$

Remark. Primitive pluriharmonic maps also come in one-parameter families [6, 9]. Hence Theorem 9 still holds when we replace M by a higher dimensional complex manifold and φ by a primitive pluriharmonic map.

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