Harmonic maps
and
Loop Groups

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I dedicate this work to Catarina, Margarida and Rafael

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Abstract

Harmonic maps from a surface M into a Riemannian symmetric space G/K correspond to certain holomorphic maps, the extended solutions, into the based loop group ΩG. Two separate classes of harmonic maps have been distinguished according to the nature of the corresponding extended solutions: harmonic maps of finite type and harmonic maps of finite uniton number.

In Chapter 3 we shall generalize Ohnita and Udagawa's results [33] concerning homogeneous projections preserving finite type property. The starting point of our study in Chapter 4 is a question: do flag transforms preserve finite type property? We shall see that this can be solved affirmatively only in some special cases but not in general; we apply these results to prove a theorem which illustrates that the class of finite type harmonic maps is essentially disjoint from that of maps with finite uniton number and to settle negatively a natural conjecture concerning harmonic tori in the quaternionic projective space. In Chapter 5 we shall make use of the Grassmannian theoretic point of view introduced by Segal [38] in order to study harmonic maps from a two-sphere into $\mathrm{Sp}(n)$. By using this methodology, we shall be able to deduce i) an "uniton factorization" of such maps and ii) an alternative characterization of harmonic two-spheres in $\mathbb{H}P^n$. Such a characterization can be seen as a natural generalization of Aithal's characterization of harmonic two-spheres in $\mathbb{H}P^2$ [1] which may be related to the approach of Bahy-El-Dien and Wood in [2]. Finally, in Chapter 6 we show that each single Bianchi-Bäcklund transform of a surface with constant Gauss curvature can be given by an appropriate dressing action and we relate Darboux and Bianchi-Bäcklund transformations for surfaces of constant mean curvature.

Contents

Chapter 1

Introduction

A map $\phi : M \to N$ of Riemannian manifolds is *harmonic* if it extremises the energy functional:

$$\int |\mathrm{d}\phi|^2 \mathrm{dvol}$$

on every compact subdomain of M. The harmonic maps in this Thesis go from a Riemann surface to compact Lie groups or symmetric spaces; they are, therefore, two-dimensional analogues of geodesics. Several integrable classes of surface are characterized by harmonicity of a suitable Gauss map. For example, a surface $f : M \to \mathbb{R}^3$ has constant mean curvature if and only if its Gauss map $M \to S^2$ is harmonic. Again, such a surface has constant Gauss curvature if and only if its Gauss map is harmonic with respect to the metric on M provided by the second fundamental form of f.

At the heart of the modern theory of harmonic maps from a Riemann surface to a Riemannian symmetric space is the observation that, in this setting, the harmonic map equations have a zero-curvature representation and so correspond to loops of flat connections. This fact was first exploited in the literature by Uhlenbeck in her study [44] of harmonic maps of $\mathbb{R}^2$ into a compact Lie group G. Uhlenbeck discovered that harmonic maps correspond to certain holomorphic maps, the *extended solutions*, into the based loop group ΩG. An extended solution reveals intimate properties of the corresponding harmonic maps:

The simplest situation occurs when the Fourier series associated to an extended solution has finitely many terms; the corresponding harmonic maps are said to have *finite uniton number*. All the harmonic maps in $\mathbb{C}$ which extend to the sphere $S^2 = \mathbb{C} \cup \infty$ are of this kind. Again, harmonic maps obtained via *twistor construction* have finite uniton number. The twistor theoretic construction of harmonic maps from holomorphic data accounts for all isotropic harmonic maps from a Riemann surface into a sphere or a complex projective space. In particular, all harmonic 2-spheres in S^n or $\mathbb{C}P^n$ arise in this way [14, 15, 21]. A general treatment of twistor spaces and twistor projections is given by Burstall and Rawnsley in [12]. From the point of view of loop groups, the harmonic maps produced via the twistor construction are characterized by the property that the corresponding extended solutions are fixed (pointwise) by a certain action of S^1 on ΩG (cf. [10]).

Harmonic maps of finite type correspond to extended solutions which can be obtained by integrating a pair of commuting Hamiltonian vector fields on certain finite-dimensional subspaces of loop algebras. It was shown in [9] that any non-conformal harmonic map of a two-torus into a rank one symmetric spaces G/K is of finite type.

We shall now outline the contents of each chapter of this thesis as well the main results we have obtained:

Chapter 2 is preparatory in nature: following [11] and [23], we give the precise loop theoretic formulation of harmonic maps into symmetric spaces and we introduce the notion of *primitive maps*. A map into a k-symmetric space, $k > 2$, is primitive if it satisfies a first order equation of Cauchy-Riemann type which arises from the geometry of the k-symmetric space. Primitive maps enjoy a number of interesting properties: they are harmonic maps and their harmonicity is preserved under homogeneous projections. In general, we shall say that a map ϕ into G/K is a *primitive harmonic map* if G/K is k-symmetric, with $k > 2$, and ϕ is primitive or if G/K is symmetric and ϕ is harmonic.

The notion of primitive harmonic map of finite type is introduced via extended solutions and via *extended framings*. We shall prove that these two constructions are equivalent.

Chapter 3. As we have mentioned above, in [9] the authors proved that any non-conformal harmonic map of a 2-torus into a rank one symmetric space G/K is of finite type. Burstall [7] generalized the notion of harmonic map of finite type to the case where the target manifold is a naturally reductive homogeneous space admitting a k-symmetric structure and proved that any weakly-conformal non-isotropic harmonic map ϕ of a 2-torus into

S^n or $\mathbb{C}P^n$ can be lifted to a primitive map of finite type into a certain k-symmetric space. After that, Ohnita and Udagawa [33] showed that, given a primitive harmonic map ψ of finite type of $\mathbb{C}$ into a generalized flag manifold G/H with its canonical k-symmetric structure, $\phi = p \circ \psi : \mathbb{C} \to G/K$ is also a primitive harmonic map of finite type for some choices of $K \supset H$, where $p : G/H \to G/K$ is the natural homogeneous projection over the j-symmetric space G/K. The condition on the closed subgroup K of G is satisfied in the complex projective space case. What underlies many of their algebraic computations is the existence of an isomorphism

$$\Lambda_\tau \mathfrak{g} \to \Lambda \mathfrak{g}$$

for any automorphism τ of order k in $\mathfrak{g}$ that we can construct explicitly in several cases. Thus, clarifying this matter we shall be able to arrive at the following conclusions:

- The condition on the closed subgroup K admits a nice geometrical formulation. In fact, $G^{\mathbb{C}}$ acts transitively on any generalized flag manifold G/H with parabolic subgroups as stabilizers. Each parabolic subgroup of $G^{\mathbb{C}}$ is the stabilizer of some flag

  $$V_1 \subset V_2 \subset \ldots \subset V_r = V$$

  for some representation V of $G^{\mathbb{C}}$. Hence, associated to the generalized flag manifold G/H with its canonical k-symmetric structure there is a parabolic subgroup P. If we take another parabolic subgroup Q such that $P \subset Q$, we have a new generalized flag manifold G/K with its canonical j-symmetric structure, such that $K \supset H$. When $G^{\mathbb{C}}$ is simple, the conditions on the choice of K referred to above amount to the demand that $P \subset Q$.

- S^n and the corresponding k-symmetric spaces used to build its primitive lifts are not generalized flag manifolds. However, both symmetric structures arise in a natural way from some parabolic subgroups $P \subset Q$; a general setting which includes this case is available and one can prove in a direct way that any harmonic 2-torus into S^n which is covered by a primitive map of finite type is also of finite type.

- The *canonical twistor fibrations of height* $k = 2$ preserve the finite type property. Canonical twistor fibrations were described in detail by Burstall and Rawnsley in [12].

Chapter 4. In [44], Uhlenbeck introduced an operation, called *adding a uniton* or *flag transform* in [12], of obtaining new harmonic maps from a given one. The starting point of our study in this chapter is a question: do flag transforms preserve finite type property? We shall see that this can be solved affirmatively only in some special cases but not in general. After this we shall apply our results to prove a theorem which illustrates that the class of finite type harmonic maps is essentially disjoint from that of maps with finite uniton number.

Taking account the results of Chapter 3, one can conclude that any harmonic map of a 2-torus into a sphere or a complex projective space is of finite type or of finite uniton number. Thus one can ask if this remains true for any other rank one symmetric space. We shall be able to settle negatively this conjecture in the quaternionic projective space case.

Chapter 5. Except in the original cases $G/K = S^n$ or $\mathbb{C}P^n$, the twistor construction does not produce all harmonic maps $S^2 \to G/K$. The search for a more general construction led to several special techniques (see the survey [19]), by means of which all harmonic maps $S^2 \to G_k(\mathbb{C}^n)$ could in principle at least, be constructed from "holomorphic data". The turning point in the theory of harmonic maps was the reformulation of the harmonic map equations in terms of extended solutions. In this new setting, the harmonic condition is immediately translated to holomorphic one. In [44], Uhlenbeck also showed that any harmonic map $\phi : S^2 \to G_k(\mathbb{C}^n)$ or $\mathrm{U}(n)$ can be obtained from a constant map by applying a finite number of flag transformations. Otherwise said, an extended solution corresponding to some harmonic map $\phi : S^2 \to G_k(\mathbb{C}^n)$ or $\mathrm{U}(n)$ admits a factorization into linear factors. These factors are in some sense holomorphic with respect to perturbations of the standard complex structure, where the perturbation depends on the previous factors. This construction of harmonic maps out of holomorphic data was made more explicit by J.C. Wood [46, 47] in the $G_k(\mathbb{C}^n)$ and $\mathrm{U}(n)$ cases, and by A. Bahy-El-Dien and J.C. Wood [2, 3] in the $G_2(\mathbb{R}^n)$ and $\mathbb{H}P^n$ cases.

Another approach to Uhlenbeck's work was introduced by Segal [38], using the Grassmannian model of the loop group $\Omega\mathrm{U}(n)$. In this setting, Segal gave an alternative proof of Uhlenbeck's factorization theorem and deduced in a beautiful way the classification theorem for harmonic maps $S^2 \to \mathbb{C}P^n$ due to J. Eells and J.C. Wood [21].

In this chapter, we shall make use of the Grassmannian theoretic point of view introduced by Segal [38] in order to study harmonic maps from a two-sphere into $\mathrm{Sp}(n)$. By using this methodology, we shall be able to deduce i) an "uniton factorization" of such maps and ii) an alternative characterization of harmonic two-spheres in $\mathbb{H}P^n$. Such a characterization can be seen as a natural generalization of Aithal's characterization of harmonic two-spheres in $\mathbb{H}P^2$ [1] which may be related to the approach of Bahy-El-Dien and Wood in [2].

Chapter 6. The Bianchi-Bäcklund transforms, which were introduced by Bianchi [4] for surfaces of positive constant Gauss curvature (CGC), are an extension of the Bäcklund transforms (see [20], §120) for surfaces of negative Gauss curvature. In contrast to the negative case, a Bianchi-Bäcklund transform of a real surface is in general complex. In order to obtain a new real surface with positive Gauss curvature, one has to apply two successive Bianchi-Bäcklund transforms, where the second transform has to be matched to the first in a particular way.

On the other hand, there is a very general mechanism for constructing a group action on a space of solutions. Here is the basic idea: let $\mathcal{G}$ be a group with subgroups $\mathcal{G}_1, \mathcal{G}_2$ such that $\mathcal{G}_1\mathcal{G}_2 = \mathcal{G}$ and $\mathcal{G}_1 \cap \mathcal{G}_2 = \{1\}$. Then $\mathcal{G}_2 \cong \mathcal{G}/\mathcal{G}_1$ so that we get a *dressing action* of $\mathcal{G}_1$ on $\mathcal{G}_2$. More generally, when $\mathcal{G}_1\mathcal{G}_2$ is only open in $\mathcal{G}$, one gets a local action. The case of importance to us is when the $\mathcal{G}_i$ are groups of holomorphic maps from subsets of the Riemann sphere $\mathbb{P}^1$ to a complex Lie group $G^{\mathbb{C}}$ distinguished by the location of their singularities.

In [31], A. Mahler showed in a constructive way that the classical Bianchi-Bäcklund procedure for obtaining a new real surface $\tilde{f}$ out of an old one f amounts to dressing the extended framing associated to the Gauss map of f, which is an harmonic map, by a certain dressing matrix. Her methodology can be briefly described as follows: starting with f and its transform $\tilde{f}$, it is possible to construct the corresponding extended framings; these extended framings are related by a certain *transition matrix* and one can obtain a explicit relationship between the dressing matrices and the transition matrices.

In this chapter we shall give an alternative approach to Mahler's work. Following the philosophy developed by Terng and Uhlenbeck [42], we start with certain basic elements of $\mathcal{G}_1$, the *simple factors*, for which the dressing action can be computed explicitly. We show that each single Bianchi-Bäcklund transform corresponds to the dressing action of a certain simple factor. A nice geometrical parameterization of these simple factors is available and we shall see how to relate it with the classical parameterization of

Bianchi-Bäcklund transforms. As a consequence we prove in an easier way the result announced by Mahler.

The classical Bianchi-Bäcklund transforms can not be applied directly to surfaces of constant mean curvature (CMC). However, it is a well known fact due to Bonnet [6] that, given a CGC surface, two of its parallel surfaces will be CMC surfaces and conversely, any CMC surface will have a parallel CGC surface. Then, we can define Bianchi-Bäcklund transformations for such surfaces by considering first their parallel CGC surface, applying Bianchi-Bäcklund transformations and then considering the parallel CMC surfaces to the transformed CGC surfaces.

Every CMC surface is isothermic, that is, it admits local conformal curvature line coordinates. Darboux [17] discovered a transformation of isothermic surfaces: the surface and its Darboux transform are characterized by the conditions that they have the same conformal structures and curvature lines and are the enveloping surfaces of a 2-sphere congruence. In [27], Hertrich-Jeromin and Pedit gave an alternative approach to the Darboux transforms: all Darboux transforms of a given isothermic surface are described by a Ricatti type equation. For suitable initial conditions, this equation will produce Darboux transforms of constant mean curvature out of an old constant mean curvature surface. We conclude this chapter by relating Darboux and Bianchi-Bäcklund transformations for CMC surfaces.

Chapter 2

Harmonic maps, loop groups and integrable systems

2.1 Harmonic maps

A map $\phi : (M, g) \to (N, h)$ of Riemannian manifolds is *harmonic* if it extremises the energy functional:

$$\int |\mathrm{d}\phi|^2 \mathrm{d}vol_M$$

on every compact subdomain of M. The associated Euler-Lagrange equation is

$$\tau_\phi \equiv \mathrm{trace}_g \nabla \mathrm{d}\phi = 0\,, \tag{2.1}$$

where ∇ is the connection on $T^*M \otimes \phi^{-1}TN$ induced by the Levi-Civita connections on M and N. The quantity $\tau_\phi \in C^\infty(\phi^{-1}TN)$ is called the *tension field*.

Good introductions to the general theory of harmonic maps can be found in the lecture notes of Eells and Lemaire [19] and the book of Urakawa [45].

Examples.

1. Harmonic maps $S^1 \to N$ are the closed geodesics, parameterized by arclength.

2. Holomorphic and anti-holomorphic maps between Kähler manifolds are harmonic.

3. A map $\phi = (\phi_1, \ldots, \phi_n) : M \to \mathbb{R}^n$ is a harmonic map if and only if each component ϕ_i is a harmonic function in the usual sense, that is, ϕ_i satisfies the Laplace's equation

$$\triangle \phi_i = 0\,.$$

In the case the domain is two dimensional, there are some special features to the theory. For example, the Euler-Lagrange equation (2.1) is conformally invariant for the domain metric and has a particularly simple form. Indeed, if M is a Riemann surface and z is a local complex coordinate on M, then $\phi : M \to N$ is harmonic (with respect to any Hermitian metric on M) if and only if

$$(\phi^{-1}\nabla^N)_{\frac{\partial}{\partial \bar{z}}} \phi_*(\frac{\partial}{\partial z}) = 0 \qquad (2.2)$$

where ∇^N is the Levi-Civita connection on TN. Equation (2.2) can be interpreted as saying that $\phi_*(\frac{\partial}{\partial z})$ is holomorphic with respect to the Kozsul-Malgrange holomorphic structure on $\phi^{-1}T^{\mathbb{C}}N \to M$:

Theorem 1. [30] Let $E \to M$ be a complex vector bundle over a Riemann surface M with connection ∇. Then there is a unique holomorphic structure on E compatible with ∇, that is, a local section σ of E is holomorphic if and only if $\nabla_{\bar{Z}}\sigma = 0$ for all $(1,0)$-vectors Z.

We now describe the class of target manifolds with which we shall be concerned:

2.2 Reductive Homogeneous spaces

2.2.1 The Maurer-Cartan form

Let N be a manifold on which a Lie group G acts transitively. Pick a base point $x_0 \in N$ with stabilizer subgroup K so that N is diffeomorphic to G/K. Suppose that $N = G/K$ is a reductive homogeneous space, which means that the Lie algebra $\mathfrak{g}$ of G admits an Ad_K-invariant splitting

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \,,$$

where $\mathfrak{k}$ is the Lie algebra of K. For each $x = g \cdot x_0$, the surjective map $\mathfrak{g} \to T_xN$ given by

$$\xi \mapsto \frac{d}{dt}_{|t=0} \exp t\xi \cdot x$$

has the Lie algebra $\mathrm{Ad}_g\mathfrak{k}$ as kernel and so restricts to an isomorphism $\mathrm{Ad}_g\mathfrak{m} \to T_xN$. The inverse map $\beta_x : T_xN \to \mathrm{Ad}_g\mathfrak{m}$ defines a $\mathfrak{g}$-valued 1-form β on N which, following [12], we call the *Maurer-Cartan form* of N. If N is actually the group manifold G, acting on itself by right translations, then β is just the (left) Maurer-Cartan form of G.

Notation. If $\mathfrak{l} \subset \mathfrak{g}^{\mathbb{C}}$ is an Ad_K-invariant subspace, let $[\mathfrak{l}]$ denote the subbundle of the trivial bundle $\underline{\mathfrak{g}^{\mathbb{C}}} = N \times \mathfrak{g}^{\mathbb{C}}$ defined by $[\mathfrak{l}]_{g\cdot x_0} = \mathrm{Ad}_g\mathfrak{l}$.

2.2.2 Symmetric and k-symmetric spaces

Let $\mathfrak{g}$ be a Lie algebra, $\tau : \mathfrak{g} \to \mathfrak{g}$ an automorphism of order $k \geq 2$ with fixed set $\mathfrak{k}$ and $\omega = e^{\frac{2\pi i}{k}}$ the primitive k-th root of unity. We have an eigenspace decomposition

$$\mathfrak{g}^{\mathbb{C}} = \sum_{i=0}^{k-1} \mathfrak{g}_i$$

where $\mathfrak{g}_i$ is the ω^i-eigenspace of τ. We extend this notation by defining $\mathfrak{g}_{i+kn} = \mathfrak{g}_i$, for any $n \in \mathbb{Z}$. Observe that $\mathfrak{g}_0 = \mathfrak{k}^{\mathbb{C}}$, $\overline{\mathfrak{g}_i} = \mathfrak{g}_{-i}$ and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \,,$$

for all $i, j \in \mathbb{Z}$. Setting

$$\mathfrak{m}^{\mathbb{C}} = \sum_{j \in \mathbb{Z}_k \setminus \{0\}} \mathfrak{g}_j$$

and $\mathfrak{m} = \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{g}$, we see that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, and so

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{2.3}$$

is a reductive decomposition of $\mathfrak{g}$. Moreover, when $k = 2$, $\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}_1$, so that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and, in this case, (2.3) is a symmetric decomposition of $\mathfrak{g}$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and suppose that τ exponentiates to give an order $k \geq 2$ automorphism, also denoted by τ, of G. Further, let $(G^\tau)_0 \subset K \subset G^\tau$, where $(G^\tau)_0$ is the identity component of

$$G^\tau = \{g \in G : \tau(g) = g\} \,,$$

so that K has Lie algebra $\mathfrak{k}$. Then the reductive homogeneous manifold $N = G/K$ is called a *k-symmetric space.*

In all our applications, we shall be concerned with the case where G is compact and semisimple. In this situation, the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$ is negative definite and so $-B_{\mathfrak{g}}$ provides a G-invariant metric on any reductive homogeneous space $N = G/K$. Of course, the 2-symmetric spaces equipped with this G-invariant metric are just the familiar Riemannian symmetric spaces of compact type.

2.3 The Maurer-Cartan equations

Let θ be the (left) Maurer-Cartan form of G. A simple calculation shows that θ satisfies the *Maurer-Cartan equation*

$$\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta] = 0\,. \tag{2.4}$$

Now let $\phi : M \to G$ be a map of a manifold M and set $\alpha = \phi^*\theta$. Then ϕ pulls back (2.4) to give

$$\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0\,, \tag{2.5}$$

so that α also satisfies the Maurer-Cartan equation. In fact, a partial converse of this is true:

Theorem 2. [40] *Let α be a $\mathfrak{g}$-valued 1-form on a simply-connected manifold M. Then $\alpha = \phi^*\theta$ for some map $\phi : M \to G$ if and only if α satisfies the Maurer-Cartan equations (2.5). In this case, ϕ is unique up to left translation by a constant element of G.*

Remarks.

1. If we interpret $\mathrm{d} + \alpha$ as a connection in the (trivial) principal bundle $M \times G$, then the Maurer-Cartan equation (2.5) says that the curvature of this connection is zero (i.e. $\mathrm{d} + \alpha$ is flat).
2. In the case G is a matrix group, we have $\phi^*\theta = \phi^{-1}\mathrm{d}\phi$.

2.4 Harmonic maps into reductive homogeneous spaces

Henceforth, we shall always take G to be a compact, connected and semisimple matrix Lie group, and the metrics on homogeneous G-spaces are induced from the Killing inner product on $\mathfrak{g}$.

Let $N = G/K$ be a reductive homogeneous space with base point $x_o = eK$, M a Riemann surface and $\phi : M \to N$ a smooth map. To simplify the exposition, we fix $M = \mathbb{C}$. From [36] we know that ϕ is harmonic if and only if the pull-back of the Maurer-Cartan form is co-closed:

$$\mathrm{d}^*\phi^*\beta = 0 \tag{2.6}$$

Now we take a lift $\psi : \mathbb{C} \to G$ with $\phi = \pi \circ \psi$, where $\pi : G \to G/K$ is the coset projection. Such (global) lifts always exist since we are working with a contractible domain. We call ψ a *framing of* ϕ. Set

$$\alpha = \psi^{-1}\mathrm{d}\psi\,.$$

Corresponding to the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a decomposition of α,

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}\,;$$

we also have a type decomposition $\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$ where $\alpha'_{\mathfrak{m}}$ is an $\mathfrak{m}^{\mathbb{C}}$-valued $(1,0)$-form and $\alpha''_{\mathfrak{m}}$ its complex conjugate. One may easily verify that

$$\phi^*\beta = \mathrm{Ad}_{\psi}\alpha_{\mathfrak{m}}\,.$$

So we can express the harmonic map equation in terms of α (which satisfies the Maurer-Cartan equation). In fact, (2.6) is equivalent to

$$\mathrm{d}\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}] = \mathrm{d}\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}] = 0 \tag{2.7}$$

$$\mathrm{d}\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}] = 0 \tag{2.8}$$

whenever

$$[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}} = 0 \tag{2.9}$$

(cf. [11]). Equation (2.9) certainly holds when N is a Riemannian symmetric space. Now, set

$$\alpha_\lambda = \lambda\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda^{-1}\alpha''_{\mathfrak{m}} \tag{2.10}$$

for all $\lambda \in S^1$; comparing coefficients of λ we conclude that (2.7) and (2.8) hold for α precisely when

$$\mathrm{d}\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \tag{2.11}$$

for all $\lambda \in S^1$. Otherwise said, assuming (2.9), the smooth map ϕ is harmonic if and only if $\mathrm{d} + \alpha_\lambda$ is a loop of flat connections.

Conversely, given a loop α_λ of $\mathfrak{g}$-valued 1-forms on $\mathbb{C}$ of the form (2.10) satisfying the Maurer-Cartan equation (2.11) for all $\lambda \in S^1$ and $[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}} = 0$, we can apply Theorem 2 in order to find, for each $\lambda \in S^1$, a map $\psi_\lambda : \mathbb{C} \to G$ such that

$$\psi_\lambda^{-1}\mathrm{d}\psi_\lambda = \alpha_\lambda\,.$$

Then we obtain a loop of harmonic maps associated to α_λ by setting $\phi_\lambda = \pi \circ \psi_\lambda : \mathbb{C} \to N$.

2.5 Primitive harmonic maps

Let $N = G/K$ be a k-symmetric space with automorphism τ and associated eigenspace decomposition

$$\mathfrak{g}^{\mathbb{C}} = \sum_{i=0}^{k-1} \mathfrak{g}_i \,.$$

As we have seen before, we get a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ by setting $\mathfrak{m} = \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{g}$, where

$$\mathfrak{m}^{\mathbb{C}} = \sum_{i \in \mathbb{Z}_k \setminus \{0\}} \mathfrak{g}_i \,.$$

Let $\phi : \mathbb{C} \to N$ be a smooth map and take a lift $\psi : \mathbb{C} \to G$ with $\phi = \pi \circ \psi$, where $\pi : G \to G/K$ is the coset projection. Corresponding to the reductive decomposition is a decomposition of $\alpha = \psi^{-1} \mathrm{d}\psi$,

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}} \,.$$

Let $\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$ be the type decomposition of $\alpha'_{\mathfrak{m}}$ into $(1,0)$-form and $(0,1)$-form of $\mathbb{C}$.

Definition 1. $\phi : \mathbb{C} \to N$ is *primitive* if $\alpha'_{\mathfrak{m}}$ is $\mathfrak{g}_1$-valued.

Remarks.

1. If $k \geq 3$ then any primitive map $\phi : \mathbb{C} \to N$ is harmonic with respect to all invariant metrics on N for which $[\mathfrak{g}_1]$ is isotropic (cf. [5]). In particular, a primitive map $\phi : \mathbb{C} \to N$ is harmonic with respect to the metric on N induced by the Killing form of $\mathfrak{g}$. Of course, when $k = 2$ all maps are primitive.

2. Following [11], we shall talk of *primitive harmonic maps* whenever we want to avoid treating the case of k-symmetric spaces with $k = 2$ separately, conscious of the fact that the term "primitive" (resp. "harmonic") is redundant when $k = 2$ (resp. $k > 2$).

Our principal interest in primitive maps derives from the fact that they are well-behaved with respect to homogeneous projections:

Theorem 3. [7] Let $K \subset H$ be closed subgroups of G with G/K k-symmetric and H τ-stable. As usual equip the reductive homogeneous spaces with the metric induced by the Killing inner product on $\mathfrak{g}$. Let $p : G/K \to G/H$ be the homogeneous projection. If $\varphi : \mathbb{C} \to G/K$ is a primitive map, then $\phi = p \circ \varphi : \mathbb{C} \to G/H$ is harmonic.

Suppose now that $\phi : \mathbb{C} \to N$ is a primitive harmonic map and consider the loop of 1-forms

$$\alpha_\lambda = \lambda\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda^{-1}\alpha''_{\mathfrak{m}} . \tag{2.12}$$

Since $\alpha'_{\mathfrak{m}}$ is $\mathfrak{g}_1$-valued, we may view α_λ as a $\Lambda\mathfrak{g}_\tau$-valued 1-form, where

$$\Lambda\mathfrak{g}_\tau = \{\xi : S^1 \to \mathfrak{g} \text{ (smooth)} : \tau(\xi(\lambda)) = \xi(\omega\lambda) \text{ for any } \lambda \in S^1\} . \tag{2.13}$$

Moreover, in this case $[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}}$ vanishes, and so the discussion in section 2.4 applies so that $\mathrm{d} + \alpha_\lambda$ is a loop of flat connections.

Conversely, suppose that α_λ is a loop of $\mathfrak{g}$-valued 1-forms of the form (2.12), such that

$$\mathrm{d}\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

for all $\lambda \in S^1$, and $\alpha'_{\mathfrak{m}}$ is $\mathfrak{g}_1$-valued. By Theorem 2, for each $\lambda \in S^1$ there is a map $\psi_\lambda : \mathbb{C} \to G$ such that

$$\psi_\lambda^{-1}\mathrm{d}\psi_\lambda = \alpha_\lambda$$

and then $\phi_\lambda = \pi \circ \psi_\lambda$ will be a S^1-family of primitive harmonic maps. Moreover, ψ_λ is unique up to left translation by a constant. We may choose these constants so that $\psi_\lambda(z_o)$ depends smoothly on λ for some (and hence every) $z_o \in \mathbb{C}$. Let ΛG_τ be the infinite-dimensional Lie group corresponding to the loop Lie algebra (2.13):

$$\Lambda G_\tau = \{\gamma : S^1 \to G \text{ (smooth)} : \gamma(\omega\lambda) = \tau(\gamma(\lambda)) \text{ for all } \lambda \in S^1\} .$$

Then we can define a smooth map $\Psi : \mathbb{C} \to \Lambda G_\tau$, by setting $\Psi(z)(\lambda) = \psi_\lambda(z)$. Ψ is called an *extended framing.*

Notation. Whenever F is a map with values in ΛG, we shall write $F_\lambda \equiv \mathrm{ev}_\lambda \circ F$, where $\mathrm{ev}_\lambda : \Lambda G_\tau \to G$ is given by evaluating the loop at $\lambda \in S^1$.

To summarize: any primitive harmonic map on $\mathbb{C}$ gives rise to an extended framing Ψ such that $\phi = \pi \circ \Psi_1$. Conversely, given an extended framing Ψ on $\mathbb{C}$, we get a loop of primitive harmonic maps by setting $\phi_\lambda = \pi \circ \Psi_\lambda$.

2.6 Extended framings of finite type

Let $\mathfrak{g}$ be a compact semisimple Lie algebra, $\tau : \mathfrak{g} \to \mathfrak{g}$ an automorphism of order k with fixed set $\mathfrak{k}$, $\omega = e^{\frac{2\pi i}{k}}$ the primitive k-th root of the unity, and

$$\mathfrak{g}^{\mathbb{C}} = \sum_{i=0}^{k-1} \mathfrak{g}_i$$

the corresponding eigenspace decomposition. Let $\mathfrak{t}$ be a maximal torus of $\mathfrak{k}$. Then $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k}^{\mathbb{C}}$. Fix a positive root system Δ^+. For each $X \in \mathfrak{g}$, $\mathrm{ad}X$ is skew with respect to the Killing inner product on $\mathfrak{g}$ and so has purely imaginary eigenvalues. Thus any root α associated to $\mathfrak{t}^{\mathbb{C}}$ belongs to $\sqrt{-1}\mathfrak{t}^*$ and so $\overline{\mathfrak{k}^{\alpha}} = \mathfrak{k}^{-\alpha}$, where $\mathfrak{k}^{\alpha}$ is the corresponding root space. Set

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{k}^{\alpha}\,,$$

the nilpotent subalgebra given by the positive root spaces. Hence

$$\overline{\mathfrak{n}} = \sum_{\alpha \in \Delta^+} \mathfrak{k}^{-\alpha}\,,$$

where the complex conjugation is taken with respect to the real form $\mathfrak{g}$, and we have

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}\,. \tag{2.14}$$

Fixing $\mathfrak{b} = \sqrt{-1}\mathfrak{t} \oplus \overline{\mathfrak{n}}$, which is a solvable subalgebra of $\mathfrak{k}^{\mathbb{C}}$,

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{b} \tag{2.15}$$

is an Iwasawa decomposition of $\mathfrak{k}^{\mathbb{C}}$. For more details on the structure of semisimple Lie groups and Lie algebras see [26].

Define the loop algebra

$$\Lambda\mathfrak{g}^{\mathbb{C}}_{\tau} = \{\xi : S^1 \to \mathfrak{g}^{\mathbb{C}} \text{ (smooth)} : \tau(\xi(\lambda)) = \xi(\omega\lambda) \text{ for any } \lambda \in S^1\}\,.$$

$\Lambda\mathfrak{g}_{\tau} = \{\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}_{\tau} :\ \xi : S^1 \to \mathfrak{g}\}$ is just the real form of $\Lambda\mathfrak{g}^{\mathbb{C}}_{\tau}$. A loop $\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}_{\tau}$ has a Fourier decomposition

$$\xi(\lambda) = \sum_{j \in \mathbb{Z}} \xi_j \lambda^j$$

with each $\xi_j \in \mathfrak{g}^{\mathbb{C}}$ satisfying $\tau(\xi_j) = \omega^j \xi_j$. Then $\Lambda\mathfrak{g}_{\tau} = \{\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}_{\tau} : \overline{\xi_j} = \xi_{-j}\}$.

Let $d = 1 \bmod k$. Define the finite dimensional subspace $\Lambda_{d,\tau}$ of $\Lambda\mathfrak{g}_{\tau}$ by

$$\Lambda_{d,\tau} = \{\xi = \sum_{j \in \mathbb{Z}} \xi_j \lambda^j \in \Lambda\mathfrak{g}_{\tau} : \xi_j = 0 \text{ for } |j| > d\}\,.$$

For any map $\xi : \mathbb{C} \to \Lambda_{d,\tau}$, ξ_{d-1} is $\mathfrak{k}^{\mathbb{C}}$-valued. Corresponding to decomposition (2.14) of $\mathfrak{k}^{\mathbb{C}}$, write $\eta \in \mathfrak{k}^{\mathbb{C}}$ as

$$\eta = \eta_{\mathfrak{t}^{\mathbb{C}}} + \eta_{\mathfrak{n}} + \eta_{\overline{\mathfrak{n}}}$$

and corresponding to decomposition (2.15) of $\mathfrak{k}^{\mathbb{C}}$, write $\eta \in \mathfrak{k}^{\mathbb{C}}$ as

$$\eta = \eta_{\mathfrak{k}} + \eta_{\mathfrak{b}} \,.$$

It is then easy to check that

$$(\xi_{d-1}\mathrm{d}z)_{\mathfrak{k}} = r(\xi_{d-1})\mathrm{d}z + \overline{r(\xi_{d-1})}\mathrm{d}\bar{z}$$

where $r : \mathfrak{k}^{\mathbb{C}} \to \mathfrak{k}^{\mathbb{C}}$ is given by $r(\eta) = \eta_{\mathfrak{n}} + \frac{1}{2}\eta_{\mathfrak{t}^{\mathbb{C}}}$.

Consider now vector fields X_1, X_2 on $\Lambda_{d,\tau}$ ($d = 1 \bmod k$) defined by

$$\frac{1}{2}(X_1 - iX_2)(\xi) = [\xi, \lambda\xi_d + r(\xi_{d-1})]$$

where the bracket is to be interpreted point-wise. It is shown in [11] that the X_i are complete commuting vector fields on $\Lambda_{d,\tau}$ so that, fixing an initial condition $\xi_o \in \Lambda_{d,\tau}$, we may integrate the corresponding flows to obtain a map $\xi : \mathbb{C} \to \Lambda_{d,\tau}$. Thus ξ is the unique solution of

$$\mathrm{d}\xi = [\xi, (\lambda\xi_d + r(\xi_{d-1}))\mathrm{d}z + (\lambda^{-1}\xi_{-d} + \overline{r(\xi_{d-1})})\mathrm{d}\bar{z}] \tag{2.16}$$

with $\xi(0) = \xi_o$, where z is the complex co-ordinate on $\mathbb{C}$. For a such ξ,

$$\alpha_\lambda = (\lambda\xi_d + r(\xi_{d-1}))\mathrm{d}z + (\lambda^{-1}\xi_{-d} + \overline{r(\xi_{d-1})})\mathrm{d}\bar{z}$$

is a $\Lambda_\tau\mathfrak{g}$-valued 1-form on $\mathbb{C}$, since the λ-coefficient ξ_d is $\mathfrak{g}_1$-valued, and α_λ is of the form (2.12).

Theorem 4. [11] $\mathrm{d} + \alpha_\lambda$ is a loop of flat connections.

Thus, we can integrate to get an extended framing $\Psi : \mathbb{C} \to \Lambda G_\tau$ (where G is a connected, compact and semisimple Lie group with Lie algebra $\mathfrak{g}$), unique up to left translation by a constant loop, with $\Psi_\lambda^{-1}\mathrm{d}\Psi_\lambda = \alpha_\lambda$. We call the extended framings so obtained *extended framings of finite type.* A map $\xi : \mathbb{C} \to \Lambda_{d,\tau}$ which satisfies equation (2.16) is called a *polynomial Killing field.*

2.7 Extended framings and a Symes formula

In [41], Symes introduced a scheme for integrating non-trivial commuting Hamiltonian flows on Lie algebras. The basic setting for the scheme is this: one has a Lie algebra $\mathfrak{g}$ which admits a direct sum decomposition into subalgebras: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$. Suppose there is a corresponding Lie group decomposition $G = K \times B$. Then the Hamiltonian flow through an initial condition $\xi_o \in \mathfrak{k}$ is given by $\mathrm{Ad}_{k^{-1}}\xi_o$, where k is the projection onto K via $G = K \times B$ of a suitable geodesic in G. The method of Symes also works in certain infinite-dimensional loop algebra settings:

Let G be a compact semisimple Lie group and τ an automorphism of order k whose fixed set is K. Let $K^{\mathbb{C}} = KB$ be the Iwasawa decomposition of $K^{\mathbb{C}}$ corresponding to the Iwasawa decomposition (2.15) of $\mathfrak{k}^{\mathbb{C}}$. Consider the following infinite-dimensional Lie groups :

$$\begin{aligned}
\Lambda G^{\mathbb{C}}_{\tau} &= \{\gamma : S^1 \to G^{\mathbb{C}} \text{ (smooth)} : \gamma(\omega\lambda) = \tau(\gamma(\lambda)) \text{ for all } \lambda \in S^1\} \\
\Lambda G_{\tau} &= \{\gamma \in \Lambda G^{\mathbb{C}}_{\tau} : \ \gamma : S^1 \to G\} \\
\Lambda_+ G^{\mathbb{C}}_{\tau} &= \{\gamma \in \Lambda G^{\mathbb{C}}_{\tau} : \gamma \text{ extends holomorphically to } D,\ \gamma(0) \in B\}
\end{aligned}$$

where $D = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

Examples of extended framings can be found by recourse to the following method:

First, we have an Iwasawa decomposition for twisted loop groups:

Theorem 5. [18] Multiplication $\Lambda G_\tau \times \Lambda_+ G^{\mathbb{C}}_\tau \to \Lambda G^{\mathbb{C}}_\tau$ is a diffeomorphism onto.

Next, let $\Lambda^{(1)}\mathfrak{g}^{\mathbb{C}}_\tau \subset \Lambda\mathfrak{g}^{\mathbb{C}}_\tau$ consist of those elements $\eta \in \Lambda\mathfrak{g}^{\mathbb{C}}_\tau$ of the form

$$\eta = \sum_{j \geq -1} \lambda^j \eta_j \,.$$

Fix $\eta_o \in \Lambda^{(1)}\mathfrak{g}^{\mathbb{C}}_\tau$. Using Theorem 5, we find maps $\Psi : \mathbb{C} \to \Lambda G_\tau$ and $\Psi_+ : \mathbb{C} \to \Lambda_+ G^{\mathbb{C}}_\tau$ such that

$$\exp(z\eta_o) = \Psi\Psi_+ \,. \tag{2.17}$$

Finally we have:

Theorem 6. [11] Ψ is an extended framing.

Extended framings of finite type all arise in this way (up to left multiplication by a constant loop):

Theorem 7. [11] An extended framing $\Psi : \mathbb{C} \to \Lambda G_\tau$ (with $\Psi(0) = e$) is of finite type if and only if, for some $d = 1 \bmod k$, there exists $\xi_o \in \Lambda_{d,\tau}$ such that

$$\exp(z\lambda^{d-1}\xi_o) = \Psi\Psi_+$$

for some smooth map $\Psi_+ : \mathbb{C} \to \Lambda_+ G^{\mathbb{C}}_\tau$.

2.8 Extended solutions

So far, we have been describing a rather general theory of primitive harmonic maps and we have distinguished among these a special class of maps which are obtained by solving a pair of commuting ordinary equations. Next we shall describe briefly the similar theory developed in [9] and its extension to the k-symmetric case in [7]. In these papers the main objects of study are maps into a Lie group rather than into an arbitrary k-symmetric space. Maps into k-symmetric spaces are then viewed as particular maps into G via a Cartan embedding. The same point of view is adopted by Uhlenbeck in her fundamental paper [44]. We will show in section 2.8.5 how these theories are related. In particular, we prove that extended framings of finite type and extended solutions of finite type produce the same class of primitive harmonic maps

2.8.1 Harmonic maps into Lie groups

Let $\phi : \mathbb{C} \to G$ be a map into a compact matrix Lie group. Equip G with a bi-invariant metric. We may view G as the homogeneous space $G/\{e\}$ so that ϕ is its own framing. Hence, ϕ is harmonic if and only if $\alpha = \phi^{-1}\mathrm{d}\phi$ is co-closed.

Let $\alpha = \alpha' + \alpha''$ be the type decomposition of α: thus α' is a $\mathfrak{g}^{\mathbb{C}}$-valued $(1,0)$-form and $\alpha'' = \overline{\alpha'}$. Combining (2.6) with the Maurer-Cartan equation for α, one concludes with Uhlenbeck [44] that $\phi : \mathbb{C} \to G$ is harmonic if and only if the loop of 1-forms given by

$$A_\lambda = \frac{1-\lambda}{2}\alpha' + \frac{1-\lambda^{-1}}{2}\alpha'' \tag{2.18}$$

satisfies the Maurer-Cartan equation for each $\lambda \in S^1$. Observe that $A_1 = 0$ and so the loop A_λ may be viewed as a 1-form with values in the based loop algebra $\Omega\mathfrak{g}$ given by

$$\Omega\mathfrak{g} = \{\xi : S^1 \to \mathfrak{g} \text{ (smooth) } : \xi(1) = 0\}\,. \tag{2.19}$$

Conversely, suppose that A_λ is a loop of $\mathfrak{g}$-valued 1-forms of the form (2.18) such that

$$\mathrm{d}A_\lambda + \frac{1}{2}[A_\lambda \wedge A_\lambda] = 0$$

for all $\lambda \in S^1$. By Theorem 2, for each $\lambda \in S^1$ there is a map $\phi_\lambda : \mathbb{C} \to G$ such that

$$\phi_\lambda^{-1}\mathrm{d}\phi_\lambda = A_\lambda$$

and then $\phi_{-1} : \mathbb{C} \to G$ will be a harmonic map. Moreover, ϕ_λ is unique up to left translation by a constant. We may choose these constants so that $\phi_1 = e$ and $\phi_\lambda(z)$ depends smoothly on λ for every $z \in \mathbb{C}$. Let ΩG be the infinite-dimensional Lie group corresponding to the loop Lie algebra (2.19):

$$\Omega G = \{\gamma : S^1 \to G \text{ (smooth)} : \gamma(1) = e\}\,.$$

Then we can define a smooth map $\Phi : \mathbb{C} \to \Omega G$, by setting $\Phi(z)(\lambda) = \phi_\lambda(z)$. Φ is called an *extended solution.*

To summarize: any harmonic map $\phi : \mathbb{C} \to G$ gives rise to an extended solution $\Phi : \mathbb{C} \to \Omega G$ such that $\phi = \Phi_{-1}$. Conversely, given an extended solution Φ on $\mathbb{C}$, we get a harmonic map by setting $\phi = \Phi_{-1} : \mathbb{C} \to G$.

2.8.2 Extended solutions of finite type

Any loop $\xi \in \Omega\mathfrak{g}$ has a unique representation

$$\xi(\lambda) = \sum_{j\neq 0} \xi_j(\lambda^j - 1)$$

with each $\xi_j \in \mathfrak{g}^{\mathbb{C}}$ and $\xi_{-j} = \overline{\xi_j}$. Fix $d \in \mathbb{N}$ and set

$$\Omega_d = \{\xi \in \Omega\mathfrak{g} : \xi_j = 0 \text{ for } |j| > d\}\,.$$

Flat $\Omega\mathfrak{g}$-valued 1-forms of the form (2.18) on $\mathbb{C}$ can be obtained by integrating commuting flows on Ω_d:

Introduce vector fields X_1, X_2 on Ω_d by

$$\frac{1}{2}(X_1 - iX_2)(\xi) = [\xi, 2i(\lambda - 1)\xi_d]\,.$$

X_1, X_2 are shown to be a commuting pair of complete vector fields (cf. [9]). Thus, if we fix an initial condition $\xi_o \in \Omega_d$, we may simultaneously integrate the X_i to get a unique map $\xi : \mathbb{C} \to \Omega_d$ satisfying

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(\lambda - 1)\xi_d] \qquad (2.20)$$

and $\xi(0) = \xi_o$. Define an $\Omega\mathfrak{g}$-valued 1-form by

$$A_\lambda = 2i(\lambda - 1)\xi_d \mathrm{d}z - 2i(\lambda^{-1} - 1)\xi_{-d}\mathrm{d}\bar{z}\,.$$

Theorem 8. [9] $\mathrm{d} + A_\lambda$ is a loop of flat connections.

Thus, we can integrate to get an extended solution $\Phi : \mathbb{C} \to \Omega G$ (where G is a compact matrix Lie group with Lie algebra $\mathfrak{g}$), unique up to left translation by a constant loop, with $\Phi_\lambda^{-1}\mathrm{d}\Phi_\lambda = A_\lambda$. We call the extended solutions so obtained *extended solutions of finite type* and the corresponding harmonic maps $\phi = \Psi_{-1} : \mathbb{C} \to G$ are *harmonic maps of finite type.* Again, we call a map $\xi : \mathbb{C} \to \Omega_d$ a *polynomial Killing field* if ξ satisfies equation (2.20).

When is a harmonic map $\phi : \mathbb{C} \to G$ of finite type? Comparing coefficients in the differential equation for the polynomial Killing field associated to such a map gives:

$$\mathrm{d}\xi_d = [\xi_d, 2i(\xi_{1-d}\mathrm{d\bar{z}} - \xi_{\mathrm{d}-1}\mathrm{dz})]$$

which takes values in $[\xi_d, \mathfrak{g}^{\mathbb{C}}]$. Otherwise said, $\mathrm{d}\xi_d$ takes values in the tangent space at ξ_d to the $\mathrm{Ad}_{\mathrm{G}^{\mathbb{C}}}$-orbit through ξ_d. We therefore deduce from the uniqueness of solutions to ODE:

Lemma 1. $\xi_d : \mathbb{C} \to \mathfrak{g}^{\mathbb{C}}$ (and hence $\alpha'(\frac{\partial}{\partial z})$) takes values in a single $\mathrm{Ad}_{\mathrm{G}^{\mathbb{C}}}$-orbit in $\mathfrak{g}^{\mathbb{C}}$.

A partial converse of this result holds for doubly periodic harmonic maps (otherwise said, lifts to the universal cover of a harmonic 2-torus):

Theorem 9. [11] A doubly periodic harmonic map $\phi : \mathbb{C} \to G$ is of finite type if $\alpha'(\frac{\partial}{\partial z}) = \phi^{-1}\frac{\partial\phi}{\partial z}$ takes values in a single semisimple $\mathrm{Ad}_{\mathrm{G}^{\mathbb{C}}}$-orbit in $\mathfrak{g}^{\mathbb{C}}$.

This condition can be further simplified:

Theorem 10. [7] A doubly periodic harmonic map $\phi : \mathbb{C} \to G$ is of finite type if $\alpha'(\frac{\partial}{\partial z}) = \phi^{-1}\frac{\partial\phi}{\partial z}$ is semisimple on a dense subset of $\mathbb{C}$.

2.8.3 Extended solutions and a Symes formula

Consider the following infinite-dimensional Lie groups:

$$\begin{aligned}
\Lambda G^{\mathbb{C}} &= \{\gamma : S^1 \to G^{\mathbb{C}} \text{ (smooth)}\} \\
\Omega G &= \{\gamma \in \Lambda G^{\mathbb{C}} : \ \gamma : S^1 \to G \text{ and } \gamma(1) = e\} \\
\Lambda_+ G^{\mathbb{C}} &= \{\gamma \in \Lambda G^{\mathbb{C}} : \gamma \text{ extends holomorphically to } D\}
\end{aligned}$$

where $D = \{\lambda \in \mathbb{C} : \ |\lambda| < 1\}$.

Examples of extended solutions can be found by recourse to the following method:

First we have an Iwasawa decomposition:

Theorem 11. [35] Multiplication $\Omega G \times \Lambda_+ G^{\mathbb{C}} \to \Lambda G^{\mathbb{C}}$ is a diffeomorphism onto.

Next, fix $\eta_o \in \Lambda \mathfrak{g}^{\mathbb{C}}$ such that $\lambda \eta_o \in \Lambda_+ \mathfrak{g}^{\mathbb{C}}$ (i.e. $\lambda \eta_o$ extends holomorphically to D). Using Theorem 11, we find maps $\Phi : \mathbb{C} \to \Omega G$ and $\Phi_+ : \mathbb{C} \to \Lambda_+ G^{\mathbb{C}}$ such that $\exp(z\eta_o) = \Phi \Phi_+$. Then we have:

Theorem 12. [11] Φ is an extended solution.

Again, extended solutions of finite type all arise via a Symes formula (up to left multiplication by a constant loop):

Theorem 13. [11] An extended solution $\Phi : \mathbb{C} \to \Omega G$ (with $\Phi(0) = e$) is of finite type if and only if, for some $d = 1 \bmod k$, there exists $\xi_o \in \Omega_d$ such that

$$\exp(z\lambda^{d-1}\xi_o) = \Phi \Phi_+$$

for some smooth map $\Phi_+ : \mathbb{C} \to \Lambda_+ G^{\mathbb{C}}$.

2.8.4 Maps into k-symmetric spaces and Cartan embeddings

Let G be a connected, compact and semisimple matrix Lie group. Let G/K be a k-symmetric space with base point $x_o = eK$, automorphism τ and Maurer-Cartan form β. As usual $\omega = e^{\frac{2\pi i}{k}}$. Define a map $\iota : G/K \to G$ by

$$\iota(g \cdot x_o) = \tau(g) g^{-1} .$$

Lemma 2. [7] Let θ be the (left) Maurer-Cartan form of G. Then, for $x \in G/K$,

$$\iota^* \theta_x = \tau_x \beta_x - \beta_x ,$$

where $\tau_{g \cdot x_o} = \mathrm{Ad}_g \circ \tau \circ \mathrm{Ad}_{g^{-1}}$.

We call ι the *Cartan embedding* of G/K into G.

Remarks.

1. ι is in general only a finite-to-one immersion. It is an embedding precisely when $K = G^{\tau}$.

2. When $k = 2$, the Cartan embedding ι, is well-known to be totally geodesic, so that if $\varphi : \mathbb{C} \to G/K$ is harmonic then $\iota \circ \varphi : \mathbb{C} \to G$ is also.

Let $\varphi : \mathbb{C} \to G/K$ be a primitive harmonic map into a k-symmetric space. Set $\delta = \varphi^*\beta$. The equations (2.7) for a framing of φ can be written in terms of δ as

$$\mathrm{d}\delta' - [\delta' \wedge \delta''] = \mathrm{d}\delta'' - [\delta' \wedge \delta''] = 0\,. \tag{2.21}$$

Then a short calculation shows that

$$A_\lambda = (\lambda - 1)\delta' + (\lambda^{-1} - 1)\delta'' \tag{2.22}$$

is a loop of flat 1-forms. To recover $\phi = \iota \circ \varphi$ from A_λ, observe that Lemma 2 gives

$$\phi^*\theta = (\omega - 1)\delta' + (\omega^{-1} - 1)\delta''\,, \tag{2.23}$$

whence $\phi^{-1}\mathrm{d}\phi = A_\omega$.

We have seen how to produce extended solutions from commuting flows on Ω_d. This procedure give rise to primitive harmonic maps if we choose the right initial condition:

Theorem 14. [7] Let $d \in \mathbb{N}$ satisfy $d = 1 \bmod k$. Let $\xi : \mathbb{C} \to \Omega_d$ be a solution of (2.20) with $\xi(0) = \xi_o \in \Omega_d \cap \Lambda\mathfrak{g}_\tau$. Let $\Phi : \mathbb{C} \to \Omega G$ be the unique extended solution such that

$$\Phi_\lambda^{-1}\mathrm{d}\Phi_\lambda = 2i(\lambda - 1)\xi_d\mathrm{d}z - 2i(\lambda^{-1} - 1)\xi_{-d}\mathrm{d}\bar{z}\,.$$

and $\Phi(0) = e$. Then, $\Phi_\omega = \iota \circ \varphi$ for some primitive harmonic map $\varphi : \mathbb{C} \to G/K$.

Such primitive harmonic maps are called *primitive harmonic maps of finite type.*

2.8.5 Extended solutions of finite type vs. extended framings of finite type

Theorem 15. Extended framings of finite type and extended solutions of finite type produce the same class of primitive harmonic maps.

To see this, start with a primitive harmonic map of finite type $\varphi : \mathbb{C} \to G/K$. Without lost of generality, suppose that $\phi(0) = \iota\circ\varphi(0) = e$. The finite type condition means that, for some $d = 1 \bmod k$, there is $\xi_o \in \Omega_d \cap \Lambda\mathfrak{g}_\tau$ such that

$$F(z) = \exp(2iz\lambda^{d-1}\xi_o) = \Phi\Phi_+\,,$$

with $\Phi_+ : \mathbb{C} \to \Lambda_+ G^{\mathbb{C}}$, and $\Phi : \mathbb{C} \to \Omega G$ an extended solution associated to ϕ ($\phi = \Phi_\omega$). On the other hand, we may see F as a map $\mathbb{C} \to \Lambda G^{\mathbb{C}}_\tau$ and use the Iwasawa decomposition of Theorem 5 to write

$$F = \Psi \Psi_+$$

with $\Psi : \mathbb{C} \to \Lambda G_\tau$ and $\Psi_+ : \mathbb{C} \to \Lambda_+ G^{\mathbb{C}}_\tau$. By Theorem 7 the map Ψ is an extended framing of finite type; it happens that Ψ is an extended framing associated to φ, that is, $\varphi = \pi \circ \Psi_1$ (cf. [11]).

Conversely, suppose that $\varphi : \mathbb{C} \to G/K$ arise from an extended framing of finite type $\Psi : \mathbb{C} \to \Lambda G_\tau$: $\varphi = \pi \circ \Psi_1$. Once again, we can suppose without lost of generality that $\Psi(0) = e$.

Lemma 3. $\Phi = \Psi \Psi_1^{-1}$ is an extended solution associated to $\phi = \iota \circ \varphi$.

Proof. the map Φ, which has values in the based loop group ΩG, satisfies:

$$\begin{aligned} \Phi_\lambda^{-1} \mathrm{d}\Phi_\lambda &= \mathrm{Ad}_{\Psi_1}(\Psi_\lambda^{-1}\mathrm{d}\Psi_\lambda - \Psi_1^{-1}\mathrm{d}\Psi_1) \\ &= \underbrace{(\lambda - 1)\mathrm{Ad}_{\Psi_1}\alpha'_{\mathfrak{m}} + (\lambda^{-1} - 1)\mathrm{Ad}_{\Psi_1}\alpha''_{\mathfrak{m}}}_{A_\lambda}; \end{aligned} \tag{2.24}$$

moreover, A_λ is of the form (2.18) and Theorem 2 ensures that $\mathrm{d} + A_\lambda$ is a loop of flat connections; whence Φ is an extended solution.

Finally note that

$$\Phi_\omega = \Psi_\omega \Psi_1^{-1} = \tau(\Psi_1)\Psi_1^{-1} = \iota \circ \pi \circ \Psi_1 = \iota \circ \varphi\,.$$

Whence Φ is an extended solution associated to $\phi = \iota \circ \varphi$.

□

Lemma 4. Let $\xi : \mathbb{C} \to \Lambda_{d,\tau}$ be the polynomial Killing field associated to Ψ, with $d = 1 \bmod k$. Define

$$\eta = \frac{1}{2i}(\lambda^k + \lambda^{-k} - 2)\mathrm{Ad}_{\Psi_1}\xi : \mathbb{C} \to \Omega_{d+k}\,.$$

η is a polynomial Killing field associated to $\Phi = \Psi\Psi_1^{-1}$:

$$\Phi_\lambda^{-1}\mathrm{d}\Phi_\lambda = 2i(\lambda - 1)\eta_{d+k}\mathrm{d}z - 2i(\lambda^{-1} - 1)\eta_{-d-k}\mathrm{d}\bar{z}\,.$$

Proof. Consider the Fourier decomposition of ξ:

$$\xi(\lambda) = \sum_{j=-d}^{d} \xi_j \lambda^j \,.$$

The map $\xi : \mathbb{C} \to \Lambda_{d,\tau}$ satisfies

$$\mathrm{d}\xi = [\xi, \Psi_\lambda^{-1}\mathrm{d}\Psi_\lambda]$$

and

$$\Psi_\lambda^{-1}\mathrm{d}\Psi_\lambda = (\lambda\xi_d + r(\xi_{d-1}))\mathrm{d}z + (\lambda^{-1}\xi_{-d} + \overline{r(\xi_{d-1})})\mathrm{d}\bar{z}\,.$$

Then

$$\begin{aligned}
\mathrm{d}\eta &= \frac{1}{2i}(\lambda^k + \lambda^{-k} - 2)\mathrm{Ad}_{\Psi_1}(\mathrm{d}\xi - [\xi, \Psi_1^{-1}\mathrm{d}\Psi_1]) \\
&= \frac{1}{2i}(\lambda^k + \lambda^{-k} - 2)\mathrm{Ad}_{\Psi_1}[\xi, \Psi_\lambda^{-1}\mathrm{d}\Psi_\lambda - \Psi_1^{-1}\mathrm{d}\Psi_1] \\
&= [\eta, \mathrm{Ad}_{\Psi_1}(\Psi_\lambda^{-1}\mathrm{d}\Psi_\lambda - \Psi_1^{-1}\mathrm{d}\Psi_1)] \\
&= [\eta, A_\lambda]\,,
\end{aligned}$$

where $A_\lambda = (\lambda - 1)\mathrm{Ad}_{\Psi_1}\alpha'_{\mathfrak{m}} + (\lambda^{-1} - 1)\mathrm{Ad}_{\Psi_1}\alpha''_{\mathfrak{m}}$. Consider the Fourier decomposition of η:

$$\eta(\lambda) = \sum_{j=-d-k}^{d+k} \eta_j(\lambda^j - 1)\,.$$

Then

$$\mathrm{Ad}_{\Psi_1}\alpha'_{\mathfrak{m}} = \mathrm{Ad}_{\Psi_1}\xi_d = 2i\eta_{d+k}$$

and

$$\mathrm{Ad}_{\Psi_1}\alpha''_{\mathfrak{m}} = \overline{\mathrm{Ad}_{\Psi_1}\alpha'_{\mathfrak{m}}} = -2i\eta_{-d-k}\,.$$

Whence $\eta : \mathbb{C} \to \Omega_{d+k}$ satisfies equation (2.20), that is, η is a polynomial Killing field. Finally, from (2.24) we conclude that η is a polynomial Killing field associated to Φ. □

Since

$$\eta(0) = \frac{1}{2i}(\lambda^k + \lambda^{-k} - 2)\xi(0) \in \Omega_{d+k} \cap \Lambda\mathfrak{g}_\tau\,,$$

these two lemmas show us that Φ and η satisfy the hypothesis of Theorem 14; whence $\varphi : \mathbb{C} \to G/K$ is a primitive harmonic map of finite type.

2.8.6 Harmonic maps of finite uniton number

Recall [14], [21] that a harmonic map ϕ of a Riemann surface M into a sphere or complex projective space has a sequence of globally defined differentials η^i on M measuring the lack of orthogonality of iterated derivatives of the map. These differentials have the following properties:

1. η^1 is a holomorphic differential which vanishes if and only if ϕ is (weakly) conformal.
2. If $\eta^1, \ldots, \eta^{j-1}$ all vanish, then η^j is a holomorphic differential.

If all the differentials η^i vanish, the map ϕ is called *isotropic*. In this case, the harmonic map is covered by a horizontal holomorphic map into an auxiliary complex manifold Z, a *twistor space*, and the methods of Algebraic Geometry can be applied in the study of isotropic harmonic maps. Since the Riemann sphere admits no non-vanishing holomorphic differentials, any harmonic map of the sphere into S^n or $\mathbb{C}P^n$ is isotropic. These results are surveyed in [19].

A general pure Lie-theoretic treatment of *twistor construction* of harmonic maps into symmetric spaces is developed in [12]. From the point of view of loop groups, a harmonic map of a simply-connected Riemann surface into a symmetric space G/K obtained via twistor construction admits an extended solution with values in the conjugacy class of some homomorphism $\gamma : S^1 \to G$ whenever G is a compact connected matrix Lie group (see [10]). An element of such class certainly is a (Laurent) polynomial in λ. In general, if a harmonic map $\phi : M \to G$ admits an extended solution Φ with values in the subspace of (Laurent) polynomial loops in ΩG of some fixed degree we say that *ϕ has finite uniton number*. In particular, isotropic harmonic maps of a simply-connected Riemann surface into S^n or $\mathbb{C}P^n$ have finite uniton number. Any extended solution on a compact Riemann surface M can be normalized to take values in such a space of polynomial loops:

Theorem 16. [38][44] Let G be a compact connected matrix Lie group, M a compact Riemann surface and $\Phi : M \to \Omega G$ an extended solution. Then there exist some $\gamma \in \Omega G$ and some $k \geq 0$ such that $\gamma\Phi$ has values in the subspace of (Laurent) polynomial loops in ΩG of degree k:

$$(\gamma\Phi)_\lambda = \sum_{i=-k}^{k} A_i \lambda^i \, .$$

As a consequence any harmonic map $S^2 \to G$ has finite uniton number.

Chapter 3

Twistor fibrations vs. finite type

In [33], Ohnita and Udagawa showed that, given a primitive harmonic map ψ of finite type of $\mathbb{C}$ into a generalized flag manifold G/H with its canonical k-symmetric structure, $\phi = p \circ \psi : \mathbb{C} \to G/K$ is also a primitive harmonic map of finite type for some choices of $K \supset H$, where $p : G/H \to G/K$ is the natural homogeneous projection over the j-symmetric space G/K. In this chapter we generalize this result of Ohnita and Udagawa (see Theorems 18, 19 and 20) and at same time we show that the condition on the closed subgroup K admits a nice geometrical formulation.

3.1 Parabolic subalgebras

Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra with Killing form denoted by B. Given a subspace $V \subset \mathfrak{g}^{\mathbb{C}}$, we shall denote by $V^{\perp}$ the polar of V with respect to B.

Definition 2. [12]

a) A subalgebra $\mathfrak{b}$ of $\mathfrak{g}^{\mathbb{C}}$ is a *Borel subalgebra* if it is a maximal solvable subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

b) A subalgebra $\mathfrak{p}$ of $\mathfrak{g}^{\mathbb{C}}$ is a *parabolic subalgebra* if it contains a Borel subalgebra.

Remark. If $\mathfrak{b} \subset \mathfrak{g}$ is a solvable Lie algebra, then there exists a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ in terms of which all the endomorphisms ad_X, $X \in \mathfrak{g}$, are expressed by upper triangular matrices (cf.[26]). Hence we see that the polar of a parabolic subalgebra is a nilpotent subalgebra $\mathfrak{p}^\perp \subset \mathfrak{p}$.

A useful criterion for detecting parabolic subalgebras is given by Lemma (4.2) in Grothendieck's paper [22]:

Proposition 1. Let $\mathfrak{n} \subset \mathfrak{g}^{\mathbb{C}}$ be a nilpotent subalgebra such that $\mathfrak{n}^\perp$ is also a subalgebra. Then $\mathfrak{n}^\perp$ is a parabolic subalgebra with *nilradical* $\mathfrak{n}$, i.e. $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{n}^\perp$.

Thus parabolic subalgebras are subalgebras whose polar is a nilpotent subalgebra.

Suppose that $\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$ is a parabolic subalgebra. Then $\mathfrak{p}^\perp$ is a nilpotent subalgebra of $\mathfrak{p}$. $\mathfrak{p}$ makes $\mathfrak{g}^{\mathbb{C}}$ into a filtered algebra: set $\mathfrak{p}^{(0)} = \mathfrak{p}$, $\mathfrak{p}^{(1)} = \mathfrak{p}^\perp$, $\mathfrak{p}^{(i+1)} = [\mathfrak{p}^{(1)}, \mathfrak{p}^{(i)}]$ for $i \geq 1$, and $\mathfrak{p}^{(i)} = \mathfrak{p}^{(-i+1)^\perp}$ for $i < 0$. The nilpotency of $\mathfrak{p}^{(1)}$ assures us of the existence of k such that $\mathfrak{p}^{(k)} \neq \{0\}$ and $\mathfrak{p}^{(k+1)} = \{0\}$. Then

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^{(-k)} \supsetneq \ldots \supsetneq \mathfrak{p}^{(k)} \supsetneq \mathfrak{p}^{(k+1)} = \{0\}$$

and, for all i we have $\mathfrak{p}^{(i)^\perp} = \mathfrak{p}^{(-i+1)}$. Call k the *height* of $\mathfrak{p}$.

Lemma 5. $[\mathfrak{p}^{(i)}, \mathfrak{p}^{(j)}] \subset \mathfrak{p}^{(i+j)}$ for all i, j.

Proof. First note that

$$B([\mathfrak{p}, \mathfrak{p}^\perp], \mathfrak{p}) = B(\mathfrak{p}^\perp, [\mathfrak{p}, \mathfrak{p}]) = 0\,,$$

so that $[\mathfrak{p}^{(0)}, \mathfrak{p}^{(1)}] \subset \mathfrak{p}^{(1)}$. Thus by Jacobi identity we have

$$[\mathfrak{p}^{(0)}, \mathfrak{p}^{(i)}] \subset \mathfrak{p}^{(i)} \quad \text{for } i > 0\,,$$

whence $[\mathfrak{p}^{(0)}, \mathfrak{p}^{(i)^\perp}] \subset \mathfrak{p}^{(i)^\perp}$ for $i > 0$. Thus

$$[\mathfrak{p}^{(0)}, \mathfrak{p}^{(j)}] \subset \mathfrak{p}^{(j)} \quad \text{for all } \ j\,.$$

Similarly (by definition)

$$[\mathfrak{p}^{(1)}, \mathfrak{p}^{(j)}] \subset \mathfrak{p}^{(j+1)} \quad \text{for } j \geq 0\,.$$

Whence

$$B([\mathfrak{p}^{(1)}, \mathfrak{p}^{(j)^\perp}], \mathfrak{p}^{(j-1)}) = B(\mathfrak{p}^{(j)^\perp}, \mathfrak{p}^{(j)}) = 0$$

i.e. $[\mathfrak{p}^{(1)}, \mathfrak{p}^{(-j+1)}] \subset \mathfrak{p}^{(-j+2)}$. Thus

$$[\mathfrak{p}^{(1)}, \mathfrak{p}^{(j)}] \subset \mathfrak{p}^{(j+1)} \quad \text{for all } \ j\,,$$

and the Jacobi identity gives

$$[\mathfrak{p}^{(i)}, \mathfrak{p}^{(j)}] \subset \mathfrak{p}^{(i+j)} \quad \text{for all} \;\; i \geq 0, j\,.$$

Finally, for $i < 0$, observe that

$$B([\mathfrak{p}^{(i)}, \mathfrak{p}^{(j)}], \mathfrak{p}^{(-i-j+1)}) = B(\mathfrak{p}^{(i)}, \mathfrak{p}^{(-i+1)}) = 0$$

by the above since $j \geq 0$ or $-i-j+1 \geq 0$. Whence

$$[\mathfrak{p}^{(i)}, \mathfrak{p}^{(j)}] \subset \mathfrak{p}^{(i+j)} \quad \text{for all} \;\; i, j\,.$$

□

Let us now consider the additional structure given by a compact real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$.

Remark. Every semisimple complex Lie algebra has a real form which is compact [26].

We denote by $\xi \mapsto \overline{\xi}$ the complex conjugation on $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$.

A parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$ together with a compact real form $\mathfrak{g}$ give rise to a grading of $\mathfrak{g}^{\mathbb{C}}$: set $\mathfrak{q} = \overline{\mathfrak{p}}$ (which is also a parabolic subalgebra of height k) and

$$\mathfrak{g}_i = \mathfrak{p}^{(i)} \cap \mathfrak{q}^{(-i)}\,. \tag{3.1}$$

Note that the above lemma gives

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}\,.$$

On the other hand, since $\mathfrak{g}$ is compact, $(X, Y)_{\mathfrak{g}} = -B(\overline{X}, Y)$ is a Hermitian inner product on $\mathfrak{g}^{\mathbb{C}}$. Given a subspace $V \subset \mathfrak{g}^{\mathbb{C}}$, we shall denote by $V^{\perp_{\mathfrak{g}}}$ the orthogonal complement of V with respect to $(.,.)_{\mathfrak{g}}$. We may therefore Gram-Schmidt orthogonalise the flag:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^{(-k)} \supsetneq \ldots \supsetneq \mathfrak{p}^{(k)} \supsetneq \mathfrak{p}^{(k+1)} = \{0\}$$

and write:

$$\mathfrak{g}^{\mathbb{C}} = \sum_{i=-k}^{k} (\mathfrak{p}^{(i)} \cap \mathfrak{p}^{(i+1)^{\perp_{\mathfrak{g}}}})\,.$$

However,
$$\mathfrak{q}^{(-i)} = \overline{\mathfrak{p}}^{(-i)} = \overline{\mathfrak{p}^{(-i)}} = \overline{\mathfrak{p}^{(i+1)^{\perp}}} = \mathfrak{p}^{(i+1)^{\perp_{\mathfrak{g}}}}$$
whence
$$\mathfrak{g}_i = \mathfrak{p}^{(i)} \cap \mathfrak{q}^{(-i)} = \mathfrak{p}^{(i)} \cap \mathfrak{p}^{(i+1)^{\perp_{\mathfrak{g}}}}$$
and so
$$\mathfrak{g}^{\mathbb{C}} = \sum_{i=-k}^{k} \mathfrak{g}_i \,.$$

Thus we have given to $\mathfrak{g}^{\mathbb{C}}$ the structure of a graded algebra. Since $\mathfrak{g}^{\mathbb{C}}$ is semisimple (and so every derivation is an inner derivation), we conclude that there is a unique $\xi \in \mathfrak{g}^{\mathbb{C}}$ with $\mathrm{ad}\xi = i\sqrt{-1}$ on $\mathfrak{g}_i$ for all $i \in \{-k, \ldots, k\}$. We call ξ the *canonical element* of $\mathfrak{p}$ (associated to the compact real form $\mathfrak{g}$).

Remark. This construction of the canonical element ξ, due to F.Burstall [1], does not involve any choice of a root system (in contrast with that presented in [12]), and so has the advantage of demonstrating that the canonical element depends only on the parabolic algebra and the choice of a compact real form of $\mathfrak{g}^{\mathbb{C}}$.

Remark. $\mathrm{ad}\xi$ has values in $\mathfrak{g}$ when restricted to $\mathfrak{g}$. But $\mathfrak{g}$, being semisimple, has trivial center $\mathfrak{z}(\mathfrak{g}) = \{0\}$, whence $\xi \in \mathfrak{g}$. At same time, ξ centralizes $\mathfrak{h} = \mathfrak{p} \cap \bar{\mathfrak{p}} \cap \mathfrak{g}$. So ξ belongs to the centre of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{z}(\mathfrak{h}) \subset \mathfrak{g}$.

Lemma 6. $\mathfrak{p}^{(i)} = \sum_{j=i}^{k} \mathfrak{g}_j$ and $\mathfrak{q}^{(i)} = \sum_{j=-k}^{-i} \mathfrak{g}_j$ for all $i \in \{-k, \ldots, k\}$.

Proof. First note that, for all $i \in \{-k, \ldots, k\}$,
$$\mathfrak{g}_i^{\perp} = \sum_{j \neq -i} \mathfrak{g}_j \tag{3.2}$$
since the $\mathfrak{g}_i$ are eigenspaces for $\mathrm{ad}\xi$ which is skew for B.

Next, we have
$$\mathfrak{g}_i \oplus \ldots \oplus \mathfrak{g}_k = \sum_{j=i}^{k} (\mathfrak{p}^{(j)} \cap \mathfrak{q}^{(-j)}) \subset \mathfrak{p}^{(i)}$$
for all $i \in \{-k, \ldots, k\}$. Thus
$$\mathfrak{g}_{1-i} \oplus \ldots \oplus \mathfrak{g}_k \subset \mathfrak{p}^{(1-i)}$$

[1] Private communication

and taking perpendicular complements gives

$$\mathfrak{p}^{(i)} \subset (\mathfrak{g}_{1-i} \oplus \ldots \oplus \mathfrak{g}_k)^\perp = \mathfrak{g}_i \oplus \ldots \oplus \mathfrak{g}_k \subset \mathfrak{p}^{(i)} .$$

Thus

$$\mathfrak{p}^{(i)} = \sum_{j=i}^{k} \mathfrak{g}_j$$

and, similarly,

$$\mathfrak{q}^{(i)} = \sum_{j=-k}^{-i} \mathfrak{g}_j .$$

□

Remark. From this lemma it is easy to conclude that $\mathfrak{g}_1$ generates $\mathfrak{p}^\perp$ and, indeed, for $r \geq 1$,

$$\mathfrak{g}_r = [\mathfrak{g}_1, [\ldots, [\mathfrak{g}_1, \mathfrak{g}_1] \ldots]]$$

with $\mathfrak{g}_1$ appearing r times.

We will use later the following two lemmas:

Lemma 7. If $\mathfrak{g}^{\mathbb{C}}$ is simple and $\mathfrak{p}$ is a parabolic subalgebra with height k, then the centre of $\mathfrak{p}^\perp$ is just $\mathfrak{p}^{(k)}$.

Proof. The inclusion $\mathfrak{p}^{(k)} \subset \mathfrak{z}(\mathfrak{p}^\perp)$ results directly from definitions. Now, since the action of $\mathfrak{h}$ on $\mathfrak{z}(\mathfrak{p}^\perp)$ is irreducible (cf. [12], Proposition 4.3), we must have $\mathfrak{p}^{(k)} = \mathfrak{z}(\mathfrak{p}^\perp)$. □

Lemma 8. Let $\mathfrak{p} \subset \tilde{\mathfrak{p}} \subset \mathfrak{g}^{\mathbb{C}}$ be two parabolic subalgebras with heights k and $\tilde{k}$, respectively. Fix a compact real form $\mathfrak{g}$. Then with obvious notations we have:

$$\mathfrak{g}_j \subset \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \ldots \oplus \tilde{\mathfrak{g}}_j \tag{3.3}$$

for all $j \geq 0$.

Suppose now that $\mathfrak{g}^{\mathbb{C}}$ is simple. Then

$$\mathfrak{g}_{k-j} \subset \tilde{\mathfrak{g}}_{\tilde{k}} \oplus \tilde{\mathfrak{g}}_{\tilde{k}-1} \oplus \ldots \oplus \tilde{\mathfrak{g}}_{\tilde{k}-j} \tag{3.4}$$

for all $j \geq 0$.

Proof. Since $\mathfrak{p} \subset \tilde{\mathfrak{p}}$, we have $\mathfrak{p}^{(j)} \subset \tilde{\mathfrak{p}}$ and $\overline{\mathfrak{p}^{(j+1)\perp}} \subset \overline{\tilde{\mathfrak{p}}^{(j+1)\perp}}$ for all $j \geq 0$; whence

$$\mathfrak{g}_j = \mathfrak{p}^{(j)} \cap \overline{\mathfrak{p}^{(j+1)\perp}} \subset \tilde{\mathfrak{p}} \cap \overline{\tilde{\mathfrak{p}}^{(j+1)\perp}} .$$

On the other hand, using lemma 6 and equation (3.2) we see that

$$\tilde{\mathfrak{p}} \cap \overline{\tilde{\mathfrak{p}}^{(j+1)\perp}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \ldots \oplus \tilde{\mathfrak{g}}_j ;$$

whence (3.3) holds.

We shall use induction to prove (3.4). The condition $\mathfrak{p} \subset \tilde{\mathfrak{p}}$ implies that $\mathfrak{z}(\mathfrak{p}^{\perp}) \subset \mathfrak{z}(\tilde{\mathfrak{p}}^{\perp})$; whence lemma 7 gives $\mathfrak{g}_k \subset \tilde{\mathfrak{g}}_{\tilde{k}}$. Suppose now that (3.4) holds for some $j \geq 0$. If $\chi \in \mathfrak{g}_{k-j-1}$ then $[\chi, \mathfrak{g}_1]$ is a non-zero (otherwise, we would have $\chi \in \mathfrak{z}(\mathfrak{p}^{\perp}) = \mathfrak{g}_k$, since $\mathfrak{g}_1$ generates $\mathfrak{p}^{\perp}$) subspace of

$$\mathfrak{g}_{k-j} \subset \tilde{\mathfrak{g}}_{\tilde{k}} \oplus \tilde{\mathfrak{g}}_{\tilde{k}-1} \oplus \ldots \oplus \tilde{\mathfrak{g}}_{\tilde{k}-j} ,$$

which means that

$$\chi \in \tilde{\mathfrak{g}}_{\tilde{k}} \oplus \tilde{\mathfrak{g}}_{\tilde{k}-1} \oplus \ldots \oplus \tilde{\mathfrak{g}}_{\tilde{k}-j-1} ,$$

since by (3.3) $\mathfrak{g}_1 \subset \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$; thus (3.3) holds for $j+1$ and by induction we conclude that it holds for every $j \geq 0$.

□

Parabolic algebras and root systems

The relationship between parabolic subalgebras and root systems is given in the following theorem:

Theorem 17. [28] Let $\mathfrak{a}$ be a Cartan subalgebra for $\mathfrak{g}^{\mathbb{C}}$ and Δ^+ a positive root system with simple roots $\alpha_1, \ldots, \alpha_l$. For each root α denote by $\mathfrak{g}^{\alpha}$ the corresponding root space. Then:

a)
$$\mathfrak{b} = \mathfrak{a} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}^{\alpha}$$
is a Borel subalgebra and any other Borel subalgebra is conjugate to $\mathfrak{b}$;

b) each subset I of $\{1, \ldots, l\}$ defines a "height" function n_I on Δ by
$$n_I(\alpha) = \sum_{i \in I} n_i$$

for $\alpha = \sum_{i=1}^{l} n_i\alpha_i$ and then

$$\mathfrak{q}_I = \mathfrak{a} \oplus \sum_{n_I(\alpha)\geq 0} \mathfrak{g}^\alpha$$

is a parabolic subalgebra. Moreover, every parabolic subalgebra is conjugate to a $\mathfrak{q}_I$ for a unique subset I of $\{1, \ldots, l\}$.

Lemma 9. Let $\mathfrak{p} = \mathfrak{q}_I$ be a parabolic algebra. Then

$$\mathfrak{p}^{(i)} = \sum_{n_I(\alpha)\geq i} \mathfrak{g}^\alpha \tag{3.5}$$

for all i and the height k of $\mathfrak{p}$ is given by $k = \max_{\alpha\in\Delta^+}\{n_I(\alpha)\}$.

Proof. Observe that

$$\mathfrak{p}^\perp = \sum_{n_I(\alpha)\geq 1} \mathfrak{g}^\alpha \tag{3.6}$$

since

$$\mathfrak{g}^{\alpha^\perp} = \sum_{\beta\in\Delta, \beta\neq -\alpha} \mathfrak{g}^\beta\,. \tag{3.7}$$

Then it is clear that

$$\mathfrak{p}^{(i)} \subset \sum_{n_I(\alpha)\geq i} \mathfrak{g}^\alpha$$

for all $i \geq 0$. Conversely, if α is a root such that $n_I(\alpha) = r \geq i \geq 0$, we may write $\alpha = \alpha_{i_1} + \ldots + \alpha_{i_s}$ with each partial sum a root and precisely r of the i_j in I whence

$$\mathfrak{g}^\alpha = [\mathfrak{g}^{\alpha_{i_s}}, [\ldots, [\mathfrak{g}^{\alpha_{i_2}}, \mathfrak{g}^{\alpha_{i_1}}] \ldots]]\,. \tag{3.8}$$

It follows from (3.6) and (3.8) that

$$\mathfrak{p}^{(i)} \supset \sum_{n_I(\alpha)\geq i} \mathfrak{g}^\alpha$$

for all $i \geq 0$. Hence (3.5) holds for all $i \geq 0$. If $i < 0$ then

$$\mathfrak{p}^{(i)} = \mathfrak{p}^{(1-i)^\perp} = (\sum_{n_I(\alpha)\geq 1-i} \mathfrak{g}^\alpha)^\perp$$

since $1 - i > 0$; using (3.7) we can finally conclude that in fact equation (3.5) holds for all i. □

Let $\mathfrak{g}$ be the compact real form of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{p}$ a parabolic subalgebra. If $\mathfrak{t} \subset \mathfrak{g}$ is a maximal torus then $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Any root α associated to $\mathfrak{t}^{\mathbb{C}}$ belongs to $\sqrt{-1}\mathfrak{t}^*$ and so $\overline{\mathfrak{g}^{\alpha}} = \mathfrak{g}^{-\alpha}$. Fix a positive root system Δ^+ and suppose that $\mathfrak{p} = \mathfrak{q}_I$ for some suitable choice of I. Since $\mathfrak{g}_r = \mathfrak{p}^{(r)} \cap \overline{\mathfrak{p}}^{(-r)}$, it follows readily from equation (3.5) that

$$\mathfrak{g}_r = \sum_{n_I(\alpha)=r} \mathfrak{g}^{\alpha} \,. \tag{3.9}$$

3.2 Canonical $(k+1)$-symmetric structures on generalized flag manifolds

Definition 3. Let $G^{\mathbb{C}}$ be a connected semisimple complex Lie group with complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. A *parabolic subgroup* of $G^{\mathbb{C}}$ is a complex Lie subgroup which is the normaliser of a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$. A *generalized flag manifold* is a homogeneous space of the form $G^{\mathbb{C}}/P$ with P a parabolic subgroup.

The following facts are well-known (cf. [48]):

a) all parabolic subgroups are connected and a subgroup is parabolic if and only if its algebra is.

b) if G is a compact real form of $G^{\mathbb{C}}$, then G acts transitively on $G^{\mathbb{C}}/P$ so that a generalized flag manifold is diffeomorphic to the real coset space $G/G \cap P$. Further, $G \cap P$ is connected and the centraliser of a torus, while, conversely, if H is the centraliser of a torus in G, then $H = G \cap P$ for at least one parabolic subgroup P of $G^{\mathbb{C}}$.

So let $F = G^{\mathbb{C}}/P = G/H$ be a generalized flag manifold, $\mathfrak{p}$ the Lie algebra of P, k the height of $\mathfrak{p}$, ξ the canonical element of $\mathfrak{p}$ associated to the compact real form $\mathfrak{g}$ (the Lie algebra of G), and $\mathfrak{g}_i$ the $i\sqrt{-1}$-eigenspace of $\mathrm{ad}\xi$. Then the Lie algebra of $H = G \cap P$ is given by $\mathfrak{h} = \mathfrak{p} \cap \overline{\mathfrak{p}} \cap \mathfrak{g}$. Consider the (inner) $(k+1)$-automorphism $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ defined by

$$\tau = \mathrm{Ad}\exp\big(\frac{2\pi\xi}{k+1}\big)\,.$$

Denote by ω the primitive $(k+1)$-th root of the unity. The ω^i-eigenspace of τ is given by

$$\mathfrak{g}^i = \mathfrak{g}_i \oplus \mathfrak{g}_{i-(k+1)}\,,$$

in particular $\mathfrak{g}^0 = \mathfrak{h}^{\mathbb{C}}$. Since $\mathrm{ad}\xi$ has values in $\mathfrak{g}$ when restricted to $\mathfrak{g}$, τ restricts to an automorphism of $\mathfrak{g}$, which we shall also denote by τ. Hence we have in G/H a canonical $(k+1)$-symmetric structure.

3.3 Twistor fibrations giving primitive harmonic maps of finite type

The following Lemma provides the key that we will use to prove the main results of this section:

Lemma 10. Let $\mathfrak{g}$ be a Lie algebra, $\tau : \mathfrak{g} \to \mathfrak{g}$ an automorphism of order k and $\sigma : S^1 \to \mathrm{Aut}\,\mathfrak{g}$ a group homomorphism such that $\sigma(\omega) = \tau$, where ω is the primitive k-th root of the unity. Then, the map $\Gamma_\tau : \Lambda\mathfrak{g} \to \Lambda\mathfrak{g}_\tau$ given by

$$\Gamma_\tau(\gamma)(\lambda) = \sigma(\lambda)\gamma(\lambda^k)$$

is an isomorphism.

Proof. Given $\gamma \in \Lambda\mathfrak{g}$,

$$\begin{aligned}
\Gamma(\gamma)(\omega\lambda) &= \sigma(\omega\lambda)\gamma(\omega^k\lambda^k) \\
&= \sigma(\omega)\sigma(\lambda)\gamma(\lambda^k) \\
&= \tau(\sigma(\lambda)\gamma(\lambda^k)) = \tau(\Gamma(\gamma)(\lambda))\,.
\end{aligned}$$

Hence $\Gamma(\gamma) \in \Lambda\mathfrak{g}_\tau$. To see that Γ is an isomorphism note that $\sigma(\lambda^{-\frac{1}{k}})\gamma(\lambda^{\frac{1}{k}})$ does not depend on the choice for the k-th root of λ if $\gamma \in \Lambda\mathfrak{g}_\tau$. Hence we can define a map $\Gamma_\tau^{-1} : \Lambda\mathfrak{g}_\tau \to \Lambda\mathfrak{g}$ by

$$\Gamma_\tau^{-1}(\gamma)(\lambda) = \sigma(\lambda^{-\frac{1}{k}})\gamma(\lambda^{\frac{1}{k}})\,,$$

for which $\Gamma_\tau \circ \Gamma_\tau^{-1} = \Gamma_\tau^{-1} \circ \Gamma_\tau = \mathrm{Id}$. ☐

3.3.1 Twistor fibrations over k-symmetric spaces I

Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra and let $\mathfrak{p} \subset \tilde{\mathfrak{p}} \subset \mathfrak{g}^{\mathbb{C}}$ be two parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$ with heights $(k-1)$ and $(\tilde{k}-1)$, respectively. Let G/H and $G/\tilde{H}$ be the corresponding generalized flag manifolds with canonical k- and $\tilde{k}$-symmetric structures denoted by τ and $\tilde{\tau}$. Then $H \subset \tilde{H}$ and $\tilde{k} \leq k$. ω denotes the primitive k-th root of the unity and $\tilde{\omega}$ the primitive

$\tilde{k}$-th root of the unity. Associated to τ and $\tilde{\tau}$ we have two decompositions of $\mathfrak{g}^{\mathbb{C}}$ into eigenspaces:

$$\mathfrak{g}^{\mathbb{C}} = \sum_{i=0}^{k-1} \mathfrak{g}^i \,, \text{ and } \quad \mathfrak{g}^{\mathbb{C}} = \sum_{i=0}^{\tilde{k}-1} \tilde{\mathfrak{g}}^i \,,$$

respectively.

Note that, in this case, the Lie subalgebra $\mathfrak{h}$ contains a maximal torus of $\mathfrak{g}$, since H is the centralizer of a torus of G; this means that $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ admit Iwasawa decompositions which are induced from an Iwasawa decomposition of $\mathfrak{g}$.

Theorem 18. Suppose $\mathfrak{p}^{(k-1)} \subset \tilde{\mathfrak{p}}^{(\tilde{k}-1)}$. Then if $\psi : \mathbb{C} \to G/H$ is a primitive map of finite type, so is $p \circ \psi : \mathbb{C} \to G/\tilde{H}$, where $p : G/H \to G/\tilde{H}$ is the canonical homogeneous projection.

Proof. Fix a maximal torus $\mathfrak{t}$ of $\mathfrak{g}$ which is contained in $\mathfrak{h} \subset \tilde{\mathfrak{h}}$. Denote by Δ the roots in $\mathfrak{g}^{\mathbb{C}}$ associated to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ and define the following subset $S \subset \Delta$ of roots:

$\alpha \in S$ if and only if $\mathfrak{g}^{\alpha} \subset \mathfrak{p}^{\perp}$.

S is non-empty and satisfies:

(i) $S \cap -S = \emptyset$

(ii) S is closed (if $\alpha, \beta \in S$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in S$).

In fact, since $\mathfrak{p}^{\perp} = \sum_{i>0} \mathfrak{g}_i$, $\overline{\mathfrak{g}^{\alpha}} = \mathfrak{g}^{-\alpha}$ and $\overline{\mathfrak{g}_i} = \mathfrak{g}_{-i}$, we have (i); on the other hand, since $\mathfrak{p}^{\perp}$ is a subalgebra and $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}] \subset \mathfrak{g}^{\alpha+\beta}$, we have (ii).

Any subset of Δ satisfying (i) and (ii) can be extended to a positive root system [28]. Let Δ^+ be such extension of S and $\mathfrak{n}$ be the subalgebra generated by all the positive root spaces.

According to our choices, we observe that:

$$\mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^0 \cap \overline{\mathfrak{n}} = \{0\} \tag{3.10}$$

and

$$\mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^2 = \mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^3 = \ldots = \mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^{(\tilde{k}-1)} = \{0\} \,. \tag{3.11}$$

In fact, since

$$\mathfrak{g}_{1-k} = \overline{\mathfrak{p}}^{(k-1)} \subset \overline{\tilde{\mathfrak{p}}}^{(\tilde{k}-1)} = \tilde{\mathfrak{g}}_{1-\tilde{k}} \,,$$

we have $\mathfrak{g}_{1-k}\cap\tilde{\mathfrak{g}}_{j-\tilde{k}}=\{0\}$ for all $j>1$. In particular, $\mathfrak{g}_{1-k}\cap\tilde{\mathfrak{g}}_0=\{0\}$. But $\mathfrak{g}^1\cap\tilde{\mathfrak{g}}^0\cap\bar{\mathfrak{n}}=\mathfrak{g}_{1-k}\cap\tilde{\mathfrak{g}}_0$, hence (3.10) holds. On the other hand, Lemma 8 also says that $\mathfrak{g}_1\cap\tilde{\mathfrak{g}}_j=\{0\}$ for all $j>1$. So

$$\mathfrak{g}^1\cap\tilde{\mathfrak{g}}^j=(\mathfrak{g}_1\cap\tilde{\mathfrak{g}}_j)\oplus(\mathfrak{g}_{1-k}\cap\tilde{\mathfrak{g}}_{j-\tilde{k}})=\{0\}$$

for all $j>1$. Hence (3.11) also holds.

Starting with the canonical elements ξ and $\tilde{\xi}$ of $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$, respectively, we can define two loops of automorphisms $\sigma,\tilde{\sigma}:S^1\to\operatorname{Aut}\mathfrak{g}$ by

$$\sigma(\lambda=e^{i\theta})=\operatorname{Ad}\exp(\theta\xi),\ \tilde{\sigma}(\lambda=e^{i\theta})=\operatorname{Ad}\exp(\theta\tilde{\xi})\,.$$

Note that $\sigma(\omega)=\tau$ and $\tilde{\sigma}(\tilde{\omega})=\tilde{\tau}$. We have also

$$\sigma(\lambda)\mathfrak{g}_j=\operatorname{Ad}\exp(\theta\xi)\mathfrak{g}_j=\lambda^j\mathfrak{g}_j\,,$$

and in the same way

$$\tilde{\sigma}(\lambda)\tilde{\mathfrak{g}}_j=\operatorname{Ad}\exp(\theta\tilde{\xi})\tilde{\mathfrak{g}}_j=\lambda^j\tilde{\mathfrak{g}}_j\,.$$

Using Lemma 10 we find an isomorphism $\Gamma:\Lambda_\tau\mathfrak{g}\to\Lambda_{\tilde{\tau}}\mathfrak{g}$ defined by

$$\Gamma(\eta)(\lambda)=\tilde{\sigma}(\lambda)\sigma(\lambda^{-\frac{\tilde{k}}{k}})\eta(\lambda^{\frac{\tilde{k}}{k}})\,.$$

We shall also denote by $\Gamma:\Lambda_\tau G\to\Lambda_{\tilde{\tau}}G$ the corresponding isomorphism between loop Lie groups. Set $\nu(\lambda)=\tilde{\sigma}(\lambda)\sigma(\lambda^{-\frac{\tilde{k}}{k}})$.

We have:

$$\nu(\lambda)=\begin{cases}\lambda^{j-i\frac{\tilde{k}}{k}} & \text{on } \mathfrak{g}^i\cap\tilde{\mathfrak{g}}^j\,,\ \text{for } i,j\neq 0\\ \lambda^{-i\frac{\tilde{k}}{k}} & \text{on } \mathfrak{g}^i\cap\tilde{\mathfrak{g}}^0\cap\mathfrak{n}\,,\ \text{for } i\neq 0\\ \lambda^{\tilde{k}-i\frac{\tilde{k}}{k}} & \text{on } \mathfrak{g}^i\cap\tilde{\mathfrak{g}}^0\cap\bar{\mathfrak{n}}\,,\ \text{for } i\neq 0\\ \operatorname{Id} & \text{on } \mathfrak{g}^0\cap\tilde{\mathfrak{g}}^0\end{cases}$$

Let now $\Psi:\mathbb{C}\to\Lambda_\tau G$ be an extended framing of ψ with associated Killing field $\eta:\mathbb{C}\to\Lambda_{d,\tau}$,

$$\eta=\sum_{j=-d}^{d}\eta_j\lambda^j\,,$$

where $d=1\mod k$, for the fixed Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ and positive root system Δ^+.

For $1 \leq i \leq k-1$ we have

$$\nu(\lambda)\eta_i \lambda^{i\frac{\tilde{k}}{k}} = \sum_{j=1}^{\tilde{k}-1} \lambda^j (\eta_i)_{\mathfrak{g}^i \cap \tilde{\mathfrak{g}}^j} + (\eta_i)_{\mathfrak{g}^i \cap \tilde{\mathfrak{g}}^0 \cap \mathfrak{n}} + \lambda^{\tilde{k}} (\eta_i)_{\mathfrak{g}^i \cap \tilde{\mathfrak{g}}^0 \cap \bar{\mathfrak{n}}} .$$

And this means, for $1 \leq i \leq k-1$ and $n \in \mathbb{N}$, that

$$\nu(\lambda)\eta_{i+nk}\lambda^{(i+nk)\frac{\tilde{k}}{k}} = \sum_{j=1}^{\tilde{k}-1} \lambda^{j+n\tilde{k}} (\eta_{i+nk})_{\mathfrak{g}^i \cap \tilde{\mathfrak{g}}^j} + \lambda^{n\tilde{k}} (\eta_{i+nk})_{\mathfrak{g}^i \cap \tilde{\mathfrak{g}}^0 \cap \mathfrak{n}} + \lambda^{\tilde{k}+n\tilde{k}} (\eta_{i+nk})_{\mathfrak{g}^i \cap \tilde{\mathfrak{g}}^0 \cap \bar{\mathfrak{n}}} . \quad (3.12)$$

At same time

$$\nu(\lambda)\eta_{nk}\lambda^{(nk)\frac{\tilde{k}}{k}} = \eta_{nk}\lambda^{n\tilde{k}} \quad (3.13)$$

also holds.

From (3.10),(3.11),(3.12), and (3.13) we conclude now that the top terms of $\tilde{\eta}(\lambda) = \Gamma(\eta)(\lambda)$ are

$$\lambda^{\tilde{k}N+1}\tilde{\eta}_{\tilde{k}N+1} = \lambda^{\tilde{k}N+1}(\eta_d)_{\mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^1}$$

and

$$\lambda^{\tilde{k}N}\tilde{\eta}_{\tilde{k}N} = \lambda^{\tilde{k}N}\{\eta_{d-1} + (\eta_d)_{\mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^0 \cap \mathfrak{n}} + \sum_{i=1}^{k-1} (\eta_{i+(N-1)k})_{\mathfrak{g}^i \cap \tilde{\mathfrak{g}}^0 \cap \bar{\mathfrak{n}}}\} ,$$

where $N \in \mathbb{N}$ is defined by $d = 1 + Nk$. So $\tilde{\eta} \in \Lambda_{\tilde{d},\tilde{\tau}}$, with $\tilde{d} = \frac{\tilde{k}}{k}(d-1) + 1$.

Now, defining $\tilde{\Psi} = \Gamma(\Psi)$ we have:

$$\begin{aligned}
\tilde{\Psi}^{-1}\tilde{\Psi}_z &= \nu(\lambda)\big(\Psi^{-1}(\lambda^{\frac{\tilde{k}}{k}})\Psi_z(\lambda^{\frac{\tilde{k}}{k}})\big) \\
&= \nu(\lambda)\big(\lambda^{\frac{\tilde{k}}{k}}\eta_d + r(\eta_{d-1})\big) \\
&= \lambda(\eta_d)_{\mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^1} + (\eta_d)_{\mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^0 \cap \mathfrak{n}} + r(\eta_{d-1}) \\
&= \lambda\tilde{\eta}_{\tilde{d}} + \tilde{r}(\tilde{\eta}_{\tilde{d}-1}) .
\end{aligned}$$

Then we conclude that $p \circ \psi$ is also of finite type. □

Remark. Let G/H be a generalized flag manifold with the canonical k-symmetric structure τ and $\mathfrak{p}$ the corresponding parabolic subalgebra. Pick a Cartan subalgebra and a positive root system Δ^+ so that $\mathfrak{p} = \mathfrak{q}_I$ for some subset I. Let $\mathfrak{n}$ be the subalgebra generated by the positive root spaces.

a) In [33] (Theorem 2.1) Ohnita and Udagawa prove that if K is a closed subgroup of G satisfying

i) $H \subset K$ and G/K is a compact Hermitian symmetric space

ii) the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is τ-stable

iii) $\mathfrak{g}^1 \cap \bar{\mathfrak{n}} \subset \mathfrak{m}^{\mathbb{C}}$

then the homogeneous projection $p : G/H \to G/K$ transforms primitive maps of finite type in harmonic maps of finite type. Now, compact Hermitian symmetric spaces are the only spaces which are simultaneously generalized flag manifolds and symmetric spaces (cf. [12]). So G/K is a generalized flag manifold. Since $H \subset K$, one can assume that G/K is associated to some parabolic subalgebra $\mathfrak{q}$ such that $\mathfrak{p} \subset \mathfrak{q}$. Finally, the condition $\mathfrak{g}^1 \cap \bar{\mathfrak{n}} \subset \mathfrak{m}^{\mathbb{C}}$ says that $\mathfrak{p}^{(k-1)} \subset \mathfrak{q}^{(1)}$.

b) In [33] (Theorem 3.5) Ohnita and Udagawa prove that if K is a closed subgroup of G satisfying

i) $H \subset K$ and G/K is a generalized flag manifold with the canonical j-symmetric structure τ' for some $2 < j < k$

ii) the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is τ-stable

iii) the eigenspace decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to τ',

$$\mathfrak{g}^{\mathbb{C}} = \sum_{j=0}^{r-1} \tilde{\mathfrak{g}}^j$$

with $\tilde{\mathfrak{g}}^0 = \mathfrak{k}^{\mathbb{C}}$, satisfies $\mathfrak{g}^1 \cap \bar{\mathfrak{n}} \subset \tilde{\mathfrak{g}}^1$ and $\mathfrak{g}^j \cap \tilde{\mathfrak{g}}^s = 0$ for $j = 1, \ldots, r-2$ and $s = j+1, \ldots, r-1$

then the homogeneous projection $p : G/H \to G/K$ transforms primitive maps of finite type in primitive maps of finite type. Once again, these conditions imply that G/K is associated to some parabolic subalgebra $\mathfrak{q}$ such that $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{p}^{(k-1)} \subset \mathfrak{q}^{(r-1)}$. Conversely, when G/K is a generalized flag manifold with parabolic subalgebra $\mathfrak{q}$ satisfying $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{p}^{(k-1)} \subset \mathfrak{q}^{(r-1)}$, then K satisfies Ohnita and Udagawa's conditions if $\mathfrak{g}$ is simple, or more generally, if $\mathfrak{z}(\mathfrak{p}^{\perp}) = \mathfrak{p}^{(k-1)}$ (see proof of Lemma 8).

Hence, Theorem 18 gives a slight generalization of Ohnita and Udagawa's results. Moreover, our conditions on the closed subgroup K are more geometrical and insightful.

Example 1. Fix in $\mathbb{C}^n$ the usual Hermitian inner product. Let $I = \{i_1 < \ldots < i_r = n\} \subset \{1, \ldots, n\}$ be a multi-index. A *flag of index* I is a filtration of $\mathbb{C}^n$ by subspaces V_i

$$V_1 \subset \ldots \subset V_r = \mathbb{C}^n$$

with $\dim V_j = i_j$. Then we find that

$$\mathfrak{p} = \{T \in \mathfrak{sl}(n,\mathbb{C}) : \ TV_j \subset V_j \ \forall j\}$$

is a parabolic subalgebra of $\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{su}^{\mathbb{C}}(n)$ for which

$$\mathfrak{p}^{(i)} = \{T \in \mathfrak{sl}(n,\mathbb{C}) : \ TV_j \subset V_{j-i} \ \forall j\}$$

where we set $V_j = \{0\}$ for $j \leq 0$. $\mathfrak{p}$ has height $r-1$. We may define mutually orthogonal subspaces $E_1, \ldots, E_r$ by

$$E_i = V_i \cap V_{i-1}^{\perp} .$$

Then

$$\mathfrak{p} \cap \mathfrak{su}(n) = \{T \in \mathfrak{su}(n) : \ TE_j \subset E_j \ \forall j\} \cong \mathfrak{s}(\mathfrak{u}(i_1) \times \ldots \times \mathfrak{u}(n - i_{r-1})) .$$

The corresponding generalized flag manifold is therefore

$$F_I = \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(i_1) \times \ldots \times \mathrm{U}(n - i_{r-1})) .$$

Consider a new flag of index $J = \{j_1 < \ldots < j_s = n\} \subset \{1, \ldots, n\}$,

$$W_1 \subset W_2 \subset \ldots \subset W_s = \mathbb{C}^n ,$$

with $s < r$. Suppose that for each $j \in \{1, \ldots, s\}$ there is some $i \geq j$, with $i \in \{1, \ldots, r\}$, such that

$$W_j = V_i .$$

Then we find a new parabolic subalgebra of $\mathfrak{sl}(n,\mathbb{C})$,

$$\tilde{\mathfrak{p}} = \{T \in \mathfrak{sl}(n,\mathbb{C}) : \ TW_j \subset W_j \ \forall j\}$$

with height $s-1$, for which $\mathfrak{p} \subset \tilde{\mathfrak{p}}$. The corresponding generalized flag manifold is now F_J and by Theorem 18 we conclude that a primitive map of finite type into F_I gives rise by projection to a primitive map of finite type into F_J.

Remark. F. Burstall proves in [7] that any weakly-conformal non-isotropic harmonic map from the 2-torus to $\mathbb{C}P^{n-1} = \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n-1))$ is covered by a primitive map of finite type into a certain generalized flag manifold F_I, with $I = \{i_1 < \ldots < i_n = n\}$ such that $i_1 = 1$. On the other hand, in [9] the authors proved that any non-conformal harmonic map of a 2-torus into a rank one symmetric space G/K is of finite type. Combining these results with Theorem 18 we conclude that *any non-istropic harmonic map from the 2-torus to $\mathbb{C}P^{n-1}$ is of finite type.*

3.3.2 Twistor fibrations over k-symmetric spaces II

Let $\mathfrak{p} \subset \tilde{\mathfrak{p}} \subset \mathfrak{g}^{\mathbb{C}}$ be two parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$ with heights $k > 2$ and $\tilde{k} \geq 2$, respectively. Fix a compact real form $\mathfrak{g}$. Let ξ and $\tilde{\xi}$ be the canonical elements of $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$, respectively. We can construct two automorphisms of $\mathfrak{g}^{\mathbb{C}}$

$$\tau = \operatorname{Ad}\exp\big(\frac{2\pi\xi}{k}\big),\ \tilde{\tau} = \operatorname{Ad}\exp(\frac{2\pi\tilde{\xi}}{\tilde{k}})$$

of order k and $\tilde{k}$, respectively. Associated to τ and $\tilde{\tau}$ we have two decompositions of $\mathfrak{g}^{\mathbb{C}}$ into eigenspaces:

$$\mathfrak{g}^{\mathbb{C}} = \sum_{i=0}^{k-1} \mathfrak{g}^i = \sum_{i=0}^{\tilde{k}-1} \tilde{\mathfrak{g}}^i\,,$$

where

$$\mathfrak{g}^i = \sum_{j=i \bmod k} \mathfrak{g}_j, \text{ and } \quad \tilde{\mathfrak{g}}^i = \sum_{j=i \bmod \tilde{k}} \tilde{\mathfrak{g}}_j\ .$$

Suppose that $\mathfrak{g}^{\mathbb{C}}$ is simple. Then $\mathfrak{g}_{-k} \subset \tilde{\mathfrak{g}}_{-\tilde{k}}$ and $\mathfrak{g}_k \subset \tilde{\mathfrak{g}}_{\tilde{k}}$ by Lemma 8; on the other hand,

$$\mathfrak{g}_0 = \mathfrak{p} \cap \overline{\mathfrak{p}} \subset \tilde{\mathfrak{p}} \cap \overline{\tilde{\mathfrak{p}}} = \tilde{\mathfrak{g}}_0\,.$$

Hence

$$\mathfrak{g}^0 = \mathfrak{g}_{-k} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_k \subset \tilde{\mathfrak{g}}_{-k} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_k = \tilde{\mathfrak{g}}^0\,.$$

Since $\mathfrak{g}^0$ and $\tilde{\mathfrak{g}}^0$ are closed under conjugation, there are subgroups K and $\tilde{K}$ of G, with Lie algebras $\mathfrak{k}$ and $\tilde{\mathfrak{k}}$, such that $K \subset \tilde{K}$, $\mathfrak{k}^{\mathbb{C}} = \mathfrak{g}^0$ and $\tilde{\mathfrak{k}}^{\mathbb{C}} = \tilde{\mathfrak{g}}^0$. In fact, we are providing to G/K and $G/\tilde{K}$ structures of k- and $\tilde{k}$-symmetric spaces. With these definitions, we have:

Theorem 19. Suppose that $\mathfrak{g}^{\mathbb{C}}$ is a simple Lie algebra. Then if $\psi : \mathbb{C} \to G/K$ is a primitive map of finite type, so is $p \circ \psi : \mathbb{C} \to G/\tilde{K}$, where $p : G/K \to G/\tilde{K}$ is the canonical homogeneous projection.

Remark. Observe that G/K and $G/\tilde{K}$ need no longer to be generalized flag manifolds. Hence Theorem 18 and Theorem 19 hold for different types of homogeneous projection.

Proof. Fix a maximal torus $\mathfrak{t}$ of g which is contained in $\mathfrak{k} \subset \tilde{\mathfrak{k}}$. Denote by Δ the set of roots in $\mathfrak{g}^{\mathbb{C}}$ associated to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ and define the subset $S \subset \Delta$ by:

$$\alpha \in S \text{ if and only if } \mathfrak{g}^\alpha \subset \mathfrak{p}^\perp \cap \tilde{\mathfrak{g}}_0 \text{ or } \mathfrak{g}^\alpha \subset \overline{\mathfrak{p}}^\perp \cap \tilde{\mathfrak{g}}_{-\tilde{k}}.$$

S is non-empty and satisfies:

(i) $S \cap -S = \emptyset$

(ii) S is closed.

So S can be extended to a positive root system Δ^+. Let $\mathfrak{n}$ be the subalgebra generated by all the positive root spaces.

According to our choices, note that:

$$\mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^0 \cap \overline{\mathfrak{n}} = \{0\} \tag{3.14}$$

and

$$\mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^2 = \mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^3 = \ldots = \mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^{(\tilde{k}-1)} = \{0\}\,. \tag{3.15}$$

In fact, $\mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^0 \cap \overline{\mathfrak{n}} = (\mathfrak{g}_{1-k} \cap \tilde{\mathfrak{g}}_0) \oplus (\mathfrak{g}_1 \cap \tilde{\mathfrak{g}}_{\tilde{k}})$. But $\mathfrak{g}_1 \subset \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ by Lemma 8, so $\mathfrak{g}_1 \cap \tilde{\mathfrak{g}}_{\tilde{k}} = \{0\}$; moreover, from Lemma 8 we also see that $\mathfrak{g}_{1-k} \subset \tilde{\mathfrak{g}}_{1-\tilde{k}} \oplus \tilde{\mathfrak{g}}_{\tilde{k}}$, whence $\mathfrak{g}_{1-k} \cap \tilde{\mathfrak{g}}_0 = \{0\}$, and (3.14) holds. On the other hand, again from Lemma 8 we conclude that

$$\begin{aligned} 0 &= (\mathfrak{g}_1 \cap \tilde{\mathfrak{g}}_j) \oplus (\mathfrak{g}_{1-k} \cap \tilde{\mathfrak{g}}_{j-\tilde{k}}) \\ &= (\mathfrak{g}_1 \oplus \mathfrak{g}_{1-k}) \cap (\tilde{\mathfrak{g}}_j \oplus \tilde{\mathfrak{g}}_{j-\tilde{k}}) = \mathfrak{g}^1 \cap \tilde{\mathfrak{g}}^j \end{aligned}$$

for all $j > 1$, that is, (3.15) also holds.

Define two loops of automorphisms $\sigma, \tilde{\sigma} : S^1 \to \operatorname{Aut} \mathfrak{g}$ by

$$\sigma(\lambda = e^{i\theta}) = \operatorname{Ad} \exp(\theta\xi)\,,\ \tilde{\sigma}(\lambda = e^{i\theta}) = \operatorname{Ad} \exp(\theta\tilde{\xi})\,.$$

Note that $\sigma(\omega) = \tau$ and $\tilde{\sigma}(\tilde{\omega}) = \tilde{\tau}$. We also have

$$\sigma(\lambda)\mathfrak{g}_j = \operatorname{Ad} \exp(\theta\xi)\mathfrak{g}_j = \lambda^j \mathfrak{g}_j\,,$$

and in the same way

$$\tilde{\sigma}(\lambda)\tilde{\mathfrak{g}}_j = \operatorname{Ad} \exp(\theta\tilde{\xi})\tilde{\mathfrak{g}}_j = \lambda^j \tilde{\mathfrak{g}}_j\,.$$

Using Lemma 10 we find an isomorphism $\Gamma : \Lambda_\tau \mathfrak{g} \to \Lambda_{\tilde{\tau}} \mathfrak{g}$ defined by

$$\Gamma(\eta)(\lambda) = \tilde{\sigma}(\lambda)\sigma(\lambda^{-\frac{\tilde{k}}{k}})\eta(\lambda^{\frac{\tilde{k}}{k}})\,.$$

Set $\nu(\lambda) = \tilde{\sigma}(\lambda)\sigma(\lambda^{-\frac{\tilde{k}}{k}})$.

We have:

$$\nu(\lambda)=\begin{cases}\lambda^{j-i\frac{\tilde{k}}{k}} & \text{on } \mathfrak{g}^i\cap\tilde{\mathfrak{g}}^j\,,\ \text{ for } i,j\neq 0\\ \lambda^{-i\frac{\tilde{k}}{k}} & \text{on } \mathfrak{g}^i\cap\tilde{\mathfrak{g}}^0\cap\mathfrak{n}\,,\ \text{ for } i\neq 0\\ \lambda^{\tilde{k}-i\frac{\tilde{k}}{k}} & \text{on } \mathfrak{g}^i\cap\tilde{\mathfrak{g}}^0\cap\overline{\mathfrak{n}}\,,\ \text{ for } i\neq 0\\ \mathrm{Id} & \text{on } \mathfrak{g}^0\cap\tilde{\mathfrak{g}}^0\end{cases}$$

So, exactly in the same way as in the proof of the previous theorem, we conclude that $p\circ\psi$ is also of finite type. □

Example 2. Let $V=\mathbb{R}^{2n+1}$, $(\,,\,)$ the usual product in V and $(\,,\,)^{\mathbb{C}}$ its complex bilinear extension.

Fix $r\in\mathbb{N}$ with $r<n+1$ and let $F^r(S^{2n})$ be the bundle of isotropic flags over S^{2n} with fiber

$$F^r_x(S^{2n})=\{w_1\subset\ldots\subset w_r\subset(T_xS^{2n})^{\mathbb{C}}: \text{each } w_j \text{ is an isotropic } j\text{-plane}\}.$$

Here, isotropy is with respect to the complexified metric on $(TS^{2n})^{\mathbb{C}}$. It is easy to see that $G=\mathrm{SO}(2n+1)$ acts transitively on $F^r(S^{2n})$ with stabilizers conjugate to

$$\overbrace{\mathrm{SO}(2)\times\ldots\times\mathrm{SO}(2)}^{r\text{ times}}\times\mathrm{SO}(2n-2r).$$

Fix a base point $(m,w_1\subset\ldots\subset w_r)\in F^r(S^{2n})$ with stabilizer H and let $\ell_0=\mathrm{span}_{\mathbb{R}}\{m\}$. Orthogonalise to obtain isotropic lines $\ell_1,\ldots,\ell_r$ and a real subspace ℓ_{r+1} in $(T_mS^{2n})^{\mathbb{C}}$ so that

$$V^{\mathbb{C}}=\ell_0^{\mathbb{C}}\oplus\sum_{i=1}^{r}(\ell_i\oplus\overline{\ell_i})\oplus\ell_{r+1}$$

is an orthogonal decomposition.

Take $k=2r+2$. Let ω be the usual k-th root of unity and define $Q\in\mathrm{O}(2n+1)$ by $Q=\omega^j$ on ℓ_j. Let τ be the order k automorphism of $\mathrm{SO}(2n+1)$ given by conjugation by Q. The identity component of the fixed set of τ is precisely the stabilizer H, so that $F^r(S^{2n})$ is a k-symmetric space. The associate reductive decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ is given by

$$\mathfrak{h}^{\mathbb{C}}=\sum_{i=1}^{r}(\overline{\ell_i}\wedge\ell_i)\oplus\wedge^2\ell_{r+1}$$

$$\mathfrak{m}^{\mathbb{C}}=\sum_{0\le i<j\le r+1}(\ell_i\wedge\ell_j)\oplus\sum_{0\le i\neq j\le r+1}(\overline{\ell_i}\wedge\ell_j)\oplus\sum_{0\le i<j\le r+1}(\overline{\ell_i}\wedge\overline{\ell_j}).$$

Moreover, the ω^j-eigenspace of τ is

$$\mathfrak{g}^j = \ell_0^{\mathbb{C}} \wedge \ell_j \oplus \sum_{i=1}^{r-j+1} (\overline{\ell_i} \wedge \ell_{j+i}) \oplus \sum_{i+s=j, i<s} (\ell_i \wedge \ell_s) \oplus \sum_{(i+s)=j \bmod k, 1 \leq i<s} (\overline{\ell_i} \wedge \overline{\ell_s})$$

for $j \in \{1, \ldots, r\}$, and

$$\mathfrak{g}^{r+1} = \ell_0^{\mathbb{C}} \wedge \ell_{r+1} \oplus \sum_{i+s=r+1, i<s} (\ell_i \wedge \ell_s) \oplus \sum_{(i+s)=r+1, i<s} (\overline{\ell_i} \wedge \overline{\ell_s}) .$$

For $j \in \{r+2, \ldots, 2r+1\}$ we have:

$$\mathfrak{g}^j = \overline{\mathfrak{g}^{2r+2-j}}$$

Let W be a maximal isotropic subspace of ℓ_{r+1}. So $\ell_{r+1} = W \oplus \overline{W}$.

Define $Q_S \in \mathrm{O}(2n+1)$ by $Q_S = -1$ on ℓ_j, for every $\ell_j \in \{1, \ldots, r+1\}$, and $Q_S = 1$ on $\ell_0^{\mathbb{C}}$. Let $\tilde{\tau}$ be the order 2 automorphism of $\mathrm{SO}(2n+1)$ given by conjugation by Q_S. The identity component of the fixed set of $\tilde{\tau}$ is precisely the stabilizer K of m. So that S^{2n} is a symmetric space. The associated reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ is given by $\mathfrak{k}^{\mathbb{C}} = \wedge^2 \tilde{V}$ and $\mathfrak{q}^{\mathbb{C}} = \ell_0^{\mathbb{C}} \wedge \tilde{V}$, where $\tilde{V} = \sum_{i=1}^{r} (\ell_i \oplus \overline{\ell_i}) \oplus \ell_{r+1}$.

We now describe the structure of $\mathfrak{so}(2n+1, \mathbb{C})$.
Fix a hermitian basis for $V^{\mathbb{C}}$

$$v_0, v_1, \ldots, v_r, v_{r+1}, \ldots, v_n, v_{n+1}, \ldots, v_{2n}$$

such that, $\overline{v_i} = v_{i+n}$ for each $i \in \{1, \ldots, n\}$, $\ell_i = \mathrm{span}_{\mathbb{C}}\{v_i\}$ for each $i \in \{0, \ldots, r\}$, and W is generated by $\{v_{r+1}, \ldots, v_n\}$. The subalgebra $\mathfrak{t}^{\mathbb{C}}$ generated by the vectors $H_i = \overline{v_i} \wedge v_i$, with $1 \leq i \leq n$ is a Cartan subalgebra of $\mathfrak{so}(2n+1, \mathbb{C})$.

In the dual space $(\mathfrak{t}^{\mathbb{C}})^*$ consider the dual basis $\{L_i\}$: $L_i(H_j) = \delta_{ij}$. The subset of roots

$$\Delta^+ = \{L_i + L_j\}_{i<j} \cup \{L_j - L_i\}_{i<j} \cup \{L_i\}_{1 \leq i \leq n}$$

forms a positive set of roots.

1. Associated to $L_i + L_j$ we have the root space $\langle v_i \wedge v_j \rangle$, $i < j$.
2. Associated to $L_j - L_i$ we have the root space $\langle \overline{v_i} \wedge v_j \rangle$, $i < j$.

3. Associated to L_i we have the root space $\langle v_i \wedge v_0 \rangle$, $1 \leq i \leq n$.

Define: $\alpha_0 = L_1$ and $\alpha_i = L_{i+1} - L_i$ for $i \in \{1, \ldots, n-1\}$. So $\alpha_0, \ldots, \alpha_{n-1}$ is the set of simple roots associated with Δ^+.

Define the subset $I \subset \{0, \ldots, n-1\}$ by:

$i \in I$ if $\mathfrak{g}^{\alpha_i} \subset \mathfrak{m}^{\mathbb{C}}$.

Note that:

i) $L_j = \alpha_0 + \alpha_1 + \ldots + \alpha_{j-1}$ if $1 \leq j \leq n$. So $n_I(L_j) = j$ for $1 \leq j \leq r$, and $n_I(L_j) = r + 1$ for $j > r$;

ii) $L_{i+j} - L_i = \alpha_i + \ldots + \alpha_{i+j-1}$. So $n_I(L_{i+j} - L_i) = j$ for $i + j \leq r + 1$, $n_I(L_{i+j} - L_i) = r+1-i$ for $i+j > r+1, i < r+1$, and $n_I(L_{i+j} - L_i) = 0$ for $i \geq r + 1$;

iii) $n_I(L_i + L_j) = n_I(L_i) + n_I(L_j)$.

Clearly, $\max_{\alpha \in \Delta^+}\{n_I(\alpha)\} = k$. Consider then the height k parabolic subalgebra

$$\mathfrak{p} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{n_I(\alpha) \geq 0} \mathfrak{g}^{\alpha}.$$

Let ξ be the canonical element of $\mathfrak{p}$ and $\mathfrak{g}_j$ the $\sqrt{-1}j$-eigenspace of $\mathrm{ad}\xi$. Then:

$$\begin{aligned}
\mathfrak{g}_0 &= W \wedge \overline{W} \oplus \sum_{i=1}^{r} (\overline{\ell_i} \wedge \ell_i) \\
&\vdots \\
\mathfrak{g}_j &= \ell_0^{\mathbb{C}} \wedge \ell_j \oplus \sum_{i=1}^{r-j} (\overline{\ell_i} \wedge \ell_{i+j}) \oplus \overline{\ell_{r-j+1}} \wedge W \oplus \sum_{i+s=j, i<s} (\ell_i \wedge \ell_s) \\
&\vdots \\
\mathfrak{g}_{r+1} &= \ell_0^{\mathbb{C}} \wedge W \oplus \sum_{i+s=r+1, 1i<s} (\ell_i \wedge \ell_s) \\
&\vdots \\
\mathfrak{g}_{r+j} &= \sum_{i,s \neq r+1, i+s=r+2, i<s} (\ell_i \wedge \ell_s) \oplus \ell_{j-1} \wedge W \\
&\vdots \\
\mathfrak{g}_{2r+2} &= W \wedge W.
\end{aligned}$$

So is now easy to see that

$$\tau = \mathrm{Ad}\exp(\frac{2\pi\xi}{k})\,.$$

Define the new subset $J \subset \{0, \ldots, n-1\}$ by:

$i \in J$ if $\mathfrak{g}^{\alpha_i} \subset \mathfrak{q}^{\mathbb{C}}$.

Note that:

i) $n_J(L_j) = 1$;

ii) $n_J(L_{i+j} - L_i) = 0$;

iii) $n_J(L_i + L_j) = n_J(L_i) + n_J(L_j) = 2$.

So $\max_{\alpha\in\Delta^+}\{n_J(\alpha)\} = 2$. Consider then the height 2 parabolic subalgebra

$$\tilde{\mathfrak{p}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{n_J(\alpha)\geq 0} \mathfrak{g}^{\alpha}\,.$$

Clearly $\mathfrak{p} \subset \tilde{\mathfrak{p}}$. Let $\tilde{\xi}$ be the canonical element of $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{g}}_j$ the $\sqrt{-1}j$-eigenspace of $\mathrm{ad}\tilde{\xi}$. Then:

$$\begin{aligned}
\mathfrak{g}_0 &= \sum_{1\leq i\neq j\leq r} (\overline{\ell_i} \wedge \ell_j) \oplus \sum_{i=1}^{r} (\overline{\ell_i} \wedge W) \oplus \overline{W} \wedge W \\
\mathfrak{g}_1 &= \sum_{j=1}^{r} (\ell_0^{\mathbb{C}} \wedge \ell_j) \oplus (\ell_0^{\mathbb{C}} \wedge W) \\
\mathfrak{g}_2 &= \sum_{1\leq i<j\leq r} (\ell_i \wedge \ell_j) \oplus (W \wedge W)\,.
\end{aligned}$$

Is now easy to see that

$$\tilde{\tau} = \mathrm{Ad}\exp\big(\frac{2\pi\tilde{\xi}}{2}\big)\,.$$

Thus $F^r(S^{2n})$ and S^{2n} satisfy the conditions of Theorem 19.

Remark. F. Burstall proves in [7] that any non-isotropic harmonic map from the 2-torus to S^n can be lifted to a primitive map of finite type into a certain generalized flag manifold of the form $F^r(S^n)$. On the other hand, in [9] the authors proved that any non-conformal harmonic map of a 2-torus into a rank one symmetric space G/K is of finite type. Combining these results with Theorem 19, we conclude that *any non-isotropic harmonic map of* 2*-torus into a sphere* S^n *is of finite type* (when n is odd we can view S^n as an equator of S^{n+1}). In [33], Ohnita and Udagawa prove this same result by constructing an embedding of $\mathfrak{so}(n+1)$ into $\mathfrak{su}(n+1)$ such that, for each j, the $\mathfrak{g}^j$-subspace for the k-symmetric structure of $F^r(S^n)$ is mapped into the $\mathfrak{g}^j$-subspace for the canonical k-symmetric structure on the generalized flag manifold over $\mathbb{C}P^n$. In this way the statement for the sphere case is presented as a corollary to the corresponding statement for the complex projective space. Our treatment of the sphere case is more direct and arises in a general setting.

3.3.3 Canonical twistor fibrations

Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra and $\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$ a parabolic subalgebra with height $k \geq 2$. Let $F = G/H$ be the corresponding generalized flag manifold with canonical k-symmetric structure denoted by τ:

$$\tau = \mathrm{Ad}\exp\big(\frac{2\pi\xi}{k+1}\big)\,,$$

where ξ is the canonical element of $\mathfrak{p}$. The ω^i-eigenspace associated to τ, ω denoting the primitive $(k+1)$-th root of the unity, is given by $\mathfrak{g}^i = \mathfrak{g}_i \oplus \mathfrak{g}_{i-k-1}$. At same time, we can define an inner involution on $\mathfrak{g}^{\mathbb{C}}$ by

$$\tau_\xi = \mathrm{Ad}\exp(\pi\xi)\,,$$

which induces a symmetric decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$, where

$$\mathfrak{k}^{\mathbb{C}} = \sum_{i\,\mathrm{even}} \mathfrak{g}_i\,, \text{ and } \quad \mathfrak{m}^{\mathbb{C}} = \sum_{i\,\mathrm{odd}} \mathfrak{g}_i\,.$$

Taking $K = (G)_0^{\tau_\xi}$, we get a symmetric space $N(F) = G/K$ with $H \subset K$. Let $p : F \to N(F)$ be the homogeneous projection. Following [12] we call this map the *canonical twistor fibration* associated to F.

With these definitions we have:

Theorem 20. For $k = 2$, if $\psi : \mathbb{C} \to F$ is a primitive map of finite type, so is $p \circ \psi : \mathbb{C} \to N(F)$.

Remark. In this case, F is a generalized flag manifold but in general $N(F)$ is not a generalized flag manifold (compare with Theorem 18).

Proof. Fix a maximal torus $\mathfrak{t}$ in $\mathfrak{h} \subset \mathfrak{k}$. Denote by Δ the set of roots in $\mathfrak{g}^{\mathbb{C}}$ associated to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ and define the subset $S \subset \Delta$ by:

$$\alpha \in S \text{ if and only if } \mathfrak{g}^{\alpha} \subset \overline{\mathfrak{p}}^{\perp}.$$

This set certainly satisfies:

(i) $S \cap -S = \emptyset$

(ii) S is closed.

So S can be extended to a positive root system Δ^+. Let $\mathfrak{n}$ be the subalgebra generated by all the positive root spaces.

The canonical element ξ of $\mathfrak{p}$ gives rise to a loop of automorphisms $\sigma : S^1 \to \operatorname{Aut}\mathfrak{g}$ defined by

$$\sigma(\lambda = e^{i\theta}) = \operatorname{Ad}\exp(\theta\xi)\,.$$

Note that $\sigma(\omega) = \tau$ and $\sigma(e^{i\pi}) = \tau_{\xi}$. We also have

$$\sigma(\lambda)\mathfrak{g}_j = \operatorname{Ad}\exp(\theta\xi)\mathfrak{g}_j = \lambda^j \mathfrak{g}_j\,.$$

Using Lemma 10 we find an isomorphism $\Gamma : \Lambda_{\tau}\mathfrak{g} \to \Lambda_{\tau_{\xi}}\mathfrak{g}$ defined by

$$\Gamma(\eta)(\lambda) = \sigma(\lambda)\sigma(\lambda^{-\frac{2}{3}})\eta(\lambda^{\frac{2}{3}}) = \sigma(\lambda^{\frac{1}{3}})\eta(\lambda^{\frac{2}{3}})\,.$$

Set $\nu(\lambda) = \sigma(\lambda^{\frac{1}{3}})$ independently of the choices made.

We have:

$$\nu(\lambda) = \begin{cases} \lambda^{i\frac{1}{3}} & \text{on } \mathfrak{g}^i \cap \overline{\mathfrak{n}} \\ \lambda^{\frac{i-3}{3}} & \text{on } \mathfrak{g}^i \cap \mathfrak{n} \\ \operatorname{Id} & \text{on } \mathfrak{g}^0 \end{cases}$$

Let now $\Psi : \mathbb{C} \to \Lambda_{\tau}G$ be an extended framing of ψ with associated Killing field $\eta : \mathbb{C} \to \Lambda_{d,\tau}$,

$$\eta = \sum_{j=-d}^{d} \eta_j \lambda^j\,,$$

where $d = 1 \mod 3$, for the fixed Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ and positive root system Δ^+.

For $i \in \{1, 2\}$ we have:

$$\nu(\lambda)\eta_i\lambda^{\frac{2i}{3}} = (\eta_i)_{\overline{\mathfrak{n}}}\lambda^i + (\eta_i)_{\mathfrak{n}}\lambda^{i-1}\,.$$

And this means, for $i \in \{1,2\}$ and $n \in \mathbb{N}$, that

$$\nu(\lambda)\eta_{i+n3}\lambda^{(i+n3)\frac{2}{3}} = (\eta_{i+n3})_{\bar{\mathfrak{n}}}\lambda^{i+2n} + (\eta_{i+n3})_{\mathfrak{n}}\lambda^{i-1+2n}\,. \tag{3.16}$$

At same time

$$\nu(\lambda)\eta_{3n}\lambda^{3n\frac{2}{3}} = \eta_{3n}\lambda^{2n} \tag{3.17}$$

also holds.

From (3.16) and (3.17) we conclude now that the top terms of $\tilde{\eta}(\lambda) = \Gamma(\eta)(\lambda)$ are

$$\lambda^{1+2N}\tilde{\eta}_{1+2N} = \lambda^{1+2N}(\eta_d)_{\bar{\mathfrak{n}}}$$

and

$$\lambda^{2N}\tilde{\eta}_{2N} = \lambda^{2N}\{\eta_{d-1} + (\eta_d)_{\mathfrak{n}} + (\eta_{d-2})_{\bar{\mathfrak{n}}}\}\,,$$

where $N \in \mathbb{N}$ is defined by $d = 1 + 3N$. So $\tilde{\eta} \in \Lambda_{\tilde{d},\tau_\xi}$, with $\tilde{d} = 1 + 2N$.

Now, defining $\tilde{\Psi} = \Gamma(\Psi)$ we have:

$$\begin{aligned}
\tilde{\Psi}^{-1}\tilde{\Psi}_z &= \nu(\lambda)\big(\Psi^{-1}(\lambda^{\frac{2}{3}})\Psi_z(\lambda^{\frac{2}{3}})\big)\\
&= \nu(\lambda)\big(\lambda^{\frac{2}{3}}\eta_d + r(\eta_{d-1})\big)\\
&= \lambda(\eta_d)_{\bar{\mathfrak{n}}} + (\eta_d)_{\mathfrak{n}} + r(\eta_{d-1})\\
&= \lambda\tilde{\eta}_{\tilde{d}} + \tilde{r}(\tilde{\eta}_{\tilde{d}-1})\,.
\end{aligned}$$

Then we conclude that $p \circ \psi$ is also of finite type. □

Example 3. 1. $F = \mathrm{SO}(2n+1)/\mathrm{U}(n)$ is a generalized flag manifold of height 2 and $N(F) = S^{2n}$.

2. $F = \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(1) \times \mathrm{U}(n-r-1))$ is a generalized flag manifold of height 2 and $N(F) = \mathbb{C}P^n$.

Chapter 4

Flag transforms vs. finite type

Flag transforms give a procedure for obtaining new harmonic maps from an old one. The starting point of our study in this chapter is a question: do flag transforms preserve the finite type property? We shall see that this is solved affirmatively only in some special cases but not in general. After this we shall apply our results to prove a theorem which illustrates that the class of finite type harmonic maps is essentially disjoint from that of maps with finite uniton number and to settle negatively a natural conjecture concerning harmonic 2-tori in quaternionic projective space.

Notation. $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$

4.1 Flag transforms

The following theorem, due to Uhlenbeck, gives us a general procedure for generating new extended solution from a given solution:

Theorem 21. [44] Let $\Phi : \mathbb{C} \to \Omega\mathrm{U}(n)$ be an extended solution, $\phi = \Phi_{-1} : \mathbb{C} \to \mathrm{U}(n)$ the corresponding harmonic map, and α a subbundle of $\underline{\mathbb{C}}^n = \mathbb{C} \times \mathbb{C}^n$ with Hermitian projection $\pi : \underline{\mathbb{C}}^n \to \alpha$ satisfying the *uniton conditions*

$$\begin{cases} \pi A_{\bar{z}} \pi^{\perp} = 0 \\ \pi^{\perp}(\bar{\partial}\pi + A_{\bar{z}}\pi) = 0 \end{cases} \tag{4.1}$$

where $A_{\bar{z}} = \frac{1}{2}\phi^{-1}\bar{\partial}\phi$. Then $\tilde{\Phi} : \mathbb{C} \to \Omega\mathrm{U}(n)$ given by

$$\tilde{\Phi}_{\lambda} = \Phi_{\lambda}(\pi + \lambda^{-1}\pi^{\perp})$$

is an extended solution.

This operation of obtaining new harmonic maps from a given one is called *adding a uniton* in [44] or *flag transform* in [12].

Let $G_k(\mathbb{C}^n)$ be the complex Grassmannian of k-planes in $\mathbb{C}^n$. The unitary group U(n) acts transitively on $G_k(\mathbb{C}^n)$ with stabilizers conjugate to $\mathrm{U}(k) \times \mathrm{U}(n-k)$. Fix a complex k-plane $V_o \in G_k(\mathbb{C}^n)$ with stabilizer K and let $\pi_o : \mathbb{C}^n \to \mathbb{C}^n$ be the Hermitian projection onto V_o. Let τ be the involution of U(n) given by conjugation by $Q_o = \pi_o - \pi_o{}^\perp$. The identity component of the fixed set of τ is K so that $G_k(\mathbb{C}^n)$ is a symmetric space. The associated symmetric decomposition $\mathfrak{u}(n) = \mathfrak{k} \oplus \mathfrak{m}$ is given by

$$\begin{aligned} \mathfrak{k}^{\mathbb{C}} &= \mathrm{Hom}(V_o, V_o) \oplus \mathrm{Hom}(V_o{}^\perp, V_o{}^\perp)\,, \\ \mathfrak{m}^{\mathbb{C}} &= \mathrm{Hom}(V_o, V_o{}^\perp) \oplus \mathrm{Hom}(V_o{}^\perp, V_o)\,. \end{aligned}$$

Consider on $\mathfrak{m}$ the $\mathrm{Ad_K}$-invariant inner product given by

$$(\xi, \eta) = -\frac{1}{2}\mathrm{tr}\,\xi\eta\,.$$

Then

$$\mathfrak{m}^+ = \mathrm{Hom}(V_o, V_o{}^\perp)$$

is an $\mathrm{Ad_K}$-invariant maximally isotropic subspace of $\mathfrak{m}^{\mathbb{C}}$ and $G_k(\mathbb{C}^n)$ inherits an invariant almost Hermitian structure, which means that $G_k(\mathbb{C}^n)$ is a Hermitian symmetric space. Since it is invariant, such structure is parallel for the Levi-Civita connection and so is integrable and Kähler.

Let $T \to G_k(\mathbb{C}^n)$ denote the tautological subbundle of $G_k(\mathbb{C}^n) \times \mathbb{C}^n$ whose fibre at $V \in G_k(\mathbb{C}^n)$ is V itself. Then $[\mathfrak{m}^+] = \mathrm{Hom}(T, T^\perp)$ and, as is well known, the Maurer-Cartan form of $G_k(\mathbb{C}^n)$ restricted to the bundle of $(1,0)$-vectors is the isomorphism $\beta^{(1,0)} : T^{(1,0)}G_k(\mathbb{C}^n) \to \mathrm{Hom}(T, T^\perp)$ given by

$$\beta(Z)\sigma = \pi_{T^\perp}(Z \cdot \sigma)$$

for σ a local section of T and $Z \in T^{(1,0)}G_k(\mathbb{C}^n)$. Here $\pi_{T^\perp}$ denotes Hermitian projection onto $T^\perp$.

The corresponding Cartan embedding $\iota_k : G_k(\mathbb{C}^n) \to \mathrm{U(n)}$ is given by $\iota_k(V) = Q_o(\pi_V - \pi_V{}^\perp)$, where $\pi_V : \mathbb{C}^n \to \mathbb{C}^n$ denotes the Hermitian projection onto the k-plane V.

The following theorem describe how to add a uniton to a harmonic map into a Grassmannian:

Theorem 22. [44] Suppose $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is a harmonic map and Φ an extended solution associated to $\phi = \iota_k \circ \psi$. Let α be a subbundle of $\underline{\mathbb{C}}^n = \mathbb{C} \times \mathbb{C}^n$ with Hermitian projection $\pi : \underline{\mathbb{C}}^n \to \alpha$ satisfying the *uniton conditions*

$$\begin{cases} \pi A_{\bar{z}} \pi^{\perp} = 0 \\ \pi^{\perp}(\bar{\partial}\pi + A_{\bar{z}}\pi) = 0 \\ [\phi, \pi] = 0 \end{cases} \tag{4.2}$$

where $A_{\bar{z}} = \frac{1}{2}\phi^{-1}\bar{\partial}\phi$. Then $\tilde{\Phi} : \mathbb{C} \to \Omega \mathrm{U}(n)$ given by

$$\tilde{\Phi}_\lambda = \Phi_\lambda(\pi + \lambda^{-1}\pi^{\perp})$$

is an extended solution for a harmonic map into a Grassmannian: $\tilde{\Phi}_{-1} = \iota_{\tilde{k}} \circ \tilde{\psi}$.

4.2 Harmonic maps into Grassmannians and subbundles

As in [2], [3] and [13] we shall frequently identify a smooth map $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ with the smooth complex subbundle $\underline{\psi}$ of the trivial bundle $\underline{\mathbb{C}}^n = \mathbb{C} \times \mathbb{C}^n$ given by setting the fibre at x equal to $\psi(x)$ for all $x \in \mathbb{C}$. Conversely any rank k subbundle of $\underline{\mathbb{C}}^n$ induces a map $\mathbb{C} \to G_k(\mathbb{C}^n)$

Observe that $\underline{\psi}$ is holomorphic (resp. antiholomorphic) subbundle of $\underline{\mathbb{C}}^n$ if and only if $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is a holomorphic (resp. antiholomorphic) map.

Definition 4. A rank k subbundle $\underline{\psi}$ of $\underline{\mathbb{C}}^n$ is said to be harmonic if $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is harmonic.

We denote the Hermitian projection onto a vector subbundle $\underline{\psi}$ by π_ψ and, for a pair of orthogonal subbundles $\underline{\psi}$ and $\underline{\phi}$, define vector bundle morphisms, $A'_{\psi\phi}, A''_{\psi\phi} : \underline{\psi} \to \underline{\phi}$ called the ∂- and $\bar{\partial}$-second fundamental forms of $\underline{\psi}$ in $\underline{\psi} \oplus \underline{\phi}$ by

$$A'_{\psi\phi}(v) = \pi_\phi(\partial v), \; A''_{\psi\phi}(v) = \pi_\phi(\bar{\partial} v)\,,$$

for v a smooth section of $\underline{\psi}$, $v \in C^\infty(\underline{\psi})$. Note that $A'_{\psi\phi}$ is minus the adjoint of $A''_{\phi\psi}$ (cf.[13]). Of particular importance are the second fundamental forms of $\underline{\psi}$ in $\underline{\mathbb{C}}^n$:

$$A'_\psi = A'_{\psi\psi^\perp}\,, \; A''_\psi = A''_{\psi\psi^\perp}$$

which, via the Maurer-Cartan isomorphism of $T^{(1,0)}G_k(\mathbb{C}^n)$ and $\mathrm{Hom}(T, T^{\perp})$, represent the $(1,0)$ components of the partial derivatives $\partial\psi$ and $\bar{\partial}\psi$, respectively. Hence a smooth map $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is holomorphic (resp. antiholomorphic) if and only if $A''_\psi = 0$ (resp. $A'_\psi = 0$). Moreover:

Lemma 11. If $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is a smooth map and $\phi : \mathbb{C} \to \mathrm{U(n)}$ is the corresponding map to the group, $\phi = \iota_k \circ \psi = Q_o(\pi_\psi - \pi_\psi{}^{\perp})$, then

$$\frac{1}{2}\phi^{-1}\partial\phi = -(A'_\psi + A'_{\psi^\perp}) \,. \tag{4.3}$$

Proof. Let β be the Maurer-Cartan form of $G_k(\mathbb{C}^n)$. From (2.23) we see that

$$\psi^*\beta(\frac{\partial}{\partial z}) = -\frac{1}{2}\phi^{-1}\frac{\partial\phi}{\partial z}$$

and recall that the conjugation in $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n)^{\mathbb{C}}$ with respect to the real form $\mathfrak{u}(n)$ is given by $\xi \mapsto -\xi^*$. Hence

$$\begin{aligned}
\psi^*\beta\big(\tfrac{\partial}{\partial z}\big) &= \psi^*\beta^{(1,0)}\big(\tfrac{\partial}{\partial z}\big) + \psi^*\beta^{(0,1)}\big(\tfrac{\partial}{\partial z}\big) = \psi^*\beta^{(1,0)}\big(\tfrac{\partial}{\partial z}\big) + \overline{\psi^*\beta^{(1,0)}\big(\tfrac{\partial}{\partial \bar{z}}\big)} \\
&= A'_\psi + \overline{A''_\psi} = A'_\psi + A'_{\psi^\perp} \,,
\end{aligned}$$

and we are done. □

In order to construct in this setting new harmonic maps from old, we need the following:

Proposition 2. [13] Let E, F be holomorphic bundles over a Riemann surface and $A : E \to F$ a holomorphic bundle morphism. Then there are unique holomorphic subbundles $\underline{\mathrm{Ker}}A$ and $\underline{\mathrm{Im}}A$ of E and F, respectively, that coincide with $\mathrm{Ker}A$ and $\mathrm{Im}A$ almost everywhere.

We give each subbundle of $\underline{\mathbb{C}}^n$ the connection induced from that of $\underline{\mathbb{C}}^n$ and corresponding Koszul–Malgrange holomorphic structure. Note that $\underline{\psi} \subset \underline{\phi}$ is holomorphic in $\underline{\phi}$ if and only if $A'_{(\phi\ominus\psi)\psi} = 0$ or, equivalently, if and only if $A''_{\psi(\phi\ominus\psi)} = 0$. Recall from [13] that $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is harmonic if and only if A'_ψ is holomorphic; this holds if and only if A''_ψ is anti-holomorphic. In this case, applying Proposition 2 we can define the ∂- and $\bar{\partial}$-Gauss bundles of ψ (or $\underline{\psi}$) by $G^{(1)}(\psi) = \underline{\mathrm{Im}}A'_\psi$ and $G^{(-1)}(\psi) = \underline{\mathrm{Im}}A''_\psi$, respectively. This bundles also represent harmonic maps (cf.[13]). Iterating this construction we set $G^{(0)}(\psi) = \underline{\psi}$, and for $i = 1, 2, \ldots$,

$$G^{(i)}(\psi) = G'(G^{(i-1)}(\psi)),\ G^{(-i)}(\psi) = G''(G^{-(i-1)}(\psi)) \,.$$

$G^{(i)}(\psi)$ is called *the i^{th}-Gauss bundle of* ψ. The harmonic map ψ is said to be *∂-irreducible* (resp. *$\bar{\partial}$-irreducible*) if $\operatorname{rank}\underline{\psi} = \operatorname{rank} G^{(1)}(\psi)$ (resp. $\operatorname{rank}\underline{\psi} = \operatorname{rank} G^{(-1)}(\psi)$) and *$\partial$-reducible* (resp. *$\bar{\partial}$-reducible*) otherwise. Following [2] and [3], by *the ∂-return map of ψ of order r*, with $r \geq 1$, we mean the map

$$c'_r(\psi) = A'_{G^{(r)}(\psi),\psi} \circ A'_{G^{(r-1)}(\psi)} \circ \ldots \circ A'_{G^{(1)}(\psi)} \circ A'_{\psi}\,.$$

If $c'_r(\psi)$ is non-zero for some r, the least such r is called *the isotropy order of* ψ and for this value of r, $c'_r(\psi)$ is called *the first ∂-return map of* ψ. If $c'_r(\psi)$ is zero for all r, we say that ψ is *(strongly) isotropic*. Since $c'_r(\psi)$ is holomorphic, we can also define *the image of the first ∂-return map* $\underline{\mathrm{Im}}c'_r(\psi)$.

In this setting flag transforms are described in the following way (cf.[46]): let $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ be a harmonic map, $\underline{\alpha}$ and $\underline{\beta}$ subbundles of $\underline{\psi}$ and $\underline{\psi}^{\perp}$ respectively which satisfy the *∂-replacement conditions*:

$$\begin{cases} \underline{\alpha} \text{ is a holomorphic subbundle of } \underline{\psi} \\ \underline{\beta} \text{ is a holomorphic subbundle of } \underline{\psi}^{\perp} \\ A'_{\psi}(\underline{\alpha}) \subset \underline{\beta} \text{ and } A'_{\psi^{\perp}}(\underline{\beta}) \subset \underline{\alpha} \end{cases} \tag{4.4}$$

Then the subbundle $\underline{\gamma} = \underline{\alpha} \oplus \underline{\beta}$ satisfies the uniton conditions (4.2). Moreover, all the subbundles satisfying (4.2) arise in this way. The harmonic map $\tilde{\psi}$ obtained by flag trasform from the pair $(\psi, \underline{\gamma})$ is given by $\underline{\tilde{\psi}} = (\underline{\psi} \ominus \underline{\alpha}) \oplus \underline{\beta}$. We recall now some well known examples (cf.[46]):

Examples.
1. The ∂-Gauss bundle $G^{(1)}(\psi)$ of a harmonic map ψ is obtained by flag transform from the pair $(\psi, \underline{\gamma} = \underline{\psi} \oplus G^{(1)}(\psi))$.

2. Suppose that $\underline{\alpha}$ is a holomorphic subbundle of $\underline{\mathrm{Ker}}A'_{\psi}$. Then with $\underline{\beta} = 0$ the replacement conditions (4.4) are satisfied.

Remark. All the constructions in this section admit a corresponding global construction with respect to an arbitrary Riemann surface M. In fact, we can get a global section $\mathcal{A}'_{\psi\phi}$ of the bundle $T^*_{(1,0)}M \otimes \underline{\psi}^* \otimes \underline{\phi}$ over M by setting $\mathcal{A}'_{\psi\phi} = \mathrm{d}z \otimes A'_{\psi\phi}$. Clearly $\mathcal{A}'_{\psi\phi}$ is holomorphic if and only if $A'_{\psi\phi}$ is.

4.3 Flag transforms preserving finite type

Let $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ be a harmonic map of finite type. Then we have an extended solution $\Phi : \mathbb{C} \to \Omega\mathrm{U}(n)$, an element $\eta_0 \in \Omega_d \cap \Lambda\mathfrak{u}(n)_\tau$ and a map

$\eta : \mathbb{C} \to \Omega_d$ with Fourier decomposition

$$\eta = \sum_{0<|i|\leq d} \eta_i \lambda^i$$

for some $d = 1 \bmod 2$, such that:

$$\begin{cases} \bar{\partial}\eta = [\eta, \Phi_\lambda^{-1}\bar{\partial}\Phi_\lambda] \\ \Phi_\lambda^{-1}\bar{\partial}\Phi_\lambda = 2i(1-\lambda^{-1})\eta_{-d} \\ \eta(0) = \eta_0 \\ \Phi_{-1} = \imath_k \circ \psi \equiv \phi \end{cases}$$

Moreover, by Lemma 4 we can suppose that for such η,

$$\begin{cases} \eta_i \text{ is a section of } \psi^*[\mathfrak{k}]^{\mathbb{C}} \text{ if } i \text{ is even} \\ \eta_i \text{ is a section of } \psi^*[\mathfrak{m}]^{\mathbb{C}} \text{ if } i \text{ is odd.} \end{cases} \tag{4.5}$$

In fact, we can take

$$\eta = \frac{1}{2i}(\lambda^k + \lambda^{-k} - 2)\mathrm{Ad}_{\Psi_1}\xi\,,$$

where Ψ is an extended framing associated to ψ, $\psi = \Psi_1 \cdot \psi(0)$, and $\xi : \mathbb{C} \to \Lambda_{d,\tau}$ is a polynomial Killing field; since $\xi = \sum \xi_i \lambda^i$ is twisted, we have $\xi_i \in \mathfrak{k}^{\mathbb{C}}$ if i is even and $\xi_i \in \mathfrak{m}^{\mathbb{C}}$ if i is odd.

Comparing coefficients in the differential equation for η we see that η_{-d} and η_d satisfy the differential equations:

$$\bar{\partial}\eta_{-d} = -2i[\eta_{-d+1}, \eta_{-d}]\,,\ \partial\eta_d = 2i[\eta_{d-1}, \eta_d]\,. \tag{4.6}$$

Lemma 12. The second fundamental forms A'_ψ and A''_ψ associated to a harmonic map $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ of finite type have no singular points.

Proof. In fact, suppose $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is a harmonic map of finite type. Since η_d is a section of $\psi^*[\mathfrak{m}]^{\mathbb{C}}$ and as a map into the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ has values in a single $\mathrm{Ad}_{\mathrm{GL}(n,\mathbb{C})}$-orbit (cf. Theorem 9), by equation (4.3) we have $A'_\psi = 2i\pi_\psi^\perp \eta_d \pi_\psi$ and the equality

$$\mathrm{Im}A'_\psi = \underline{\mathrm{Im}}A'_\psi \tag{4.7}$$

holds everywhere. $\square$

Now, let $\underline{\alpha}$ be a subbundle of $\underline{\mathbb{C}}^n$ with Hermitian projection π satisfying uniton conditions (4.2):

$$\begin{cases} \pi A_{\bar z}\pi^\perp = 0 \\ \pi^\perp(\bar\partial\pi + A_{\bar z}\pi) = 0 \\ [\psi,\pi] = 0 \end{cases}$$

where

$$A_{\bar z} = \frac{1}{2}\phi^{-1}\bar\partial\phi = 2i\eta_{-d}\,. \tag{4.8}$$

By Theorem 22 $\tilde\Phi_\lambda = \Phi_\lambda(\pi+\lambda^{-1}\pi^\perp)$ defines an extended solution associated to some harmonic map $\tilde\psi$ into a Grassmannian $G_{\tilde k}(\mathbb{C}^n)$. Is $\tilde\psi$ of finite type? We shall show that for some special subbundles α this question can be solved affirmatively. Start by considering $\tilde\eta = \mathrm{Ad}_{(\pi+\lambda\pi^\perp)}\eta$. This map certainly satisfies

$$\bar\partial\tilde\eta = [\tilde\eta, \tilde\Phi_\lambda^{-1}\bar\partial\tilde\Phi_\lambda]\,, \tag{4.9}$$

and using the uniton conditions we obtain

$$\begin{aligned}
\tilde\Phi_\lambda^{-1}\bar\partial\tilde\Phi_\lambda &= \mathrm{Ad}_{(\pi+\lambda\pi^\perp)}(2i(1-\lambda^{-1})\eta_{-d}) + (\pi+\lambda\pi^\perp)\bar\partial(\pi+\lambda^{-1}\pi^\perp) \\
&= (1-\lambda^{-1})\{2i(\pi\eta_{-d}\pi + \pi^\perp\eta_{-d}\pi^\perp + \lambda\pi^\perp\eta_{-d}\pi) + (\pi+\lambda\pi^\perp)\bar\partial\pi\} \\
&= (1-\lambda^{-1})\{2i(\pi\eta_{-d}\pi + \pi^\perp\eta_{-d}\pi^\perp) + \pi\bar\partial\pi\} \\
&= (1-\lambda^{-1})\{2i(\eta_{-d} - \pi^\perp\eta_{-d}\pi) + \pi\bar\partial\pi\} \\
&= (1-\lambda^{-1})\{-2i\eta_{-d} + (\pi\bar\partial\pi + \pi^\perp\bar\partial\pi)\} \\
&= 2i(1-\lambda^{-1})(\eta_{-d} - \frac{i}{2}\bar\partial\pi)\,.
\end{aligned} \tag{4.10}$$

We shall now analyse the top terms of

$$\tilde\eta = \mathrm{Ad}_{(\pi+\lambda\pi^\perp)}\eta = \sum_{0<|n|\le d} \lambda^i(\pi+\lambda\pi^\perp)\eta_i(\pi+\lambda^{-1}\pi^\perp)\,.$$

Considering the uniton conditions we see that $\tilde\eta_{-d-1} = \pi\eta_{-d}\pi^\perp = 0$, whence $\tilde\eta$ has values in Ω_d, and

$$\begin{aligned}
\tilde\eta_{-d} &= \pi^\perp\eta_{-d}\pi^\perp + \pi\eta_{-d+1}\pi^\perp + \pi\eta_{-d}\pi \\
&= \eta_{-d} - \pi^\perp\eta_{-d}\pi + \pi\eta_{-d+1}\pi^\perp \\
&= \eta_{-d} - \frac{i}{2}\pi^\perp\bar\partial\pi + \pi\eta_{-d+1}\pi^\perp \\
&= \eta_{-d} - \frac{i}{2}\bar\partial\pi + \frac{i}{2}\pi\bar\partial\pi + \pi\eta_{-d+1}\pi^\perp\,.
\end{aligned} \tag{4.11}$$

Moreover, since $[\phi, \pi] = 0$, the new harmonic map into the group $\tilde{\phi} = \imath_{\tilde{k}} \circ \tilde{\psi}$ also commutes with π, $[\tilde{\phi}, \pi] = 0$, and from this one can easily see that $\tilde{\eta}$ satisfies the right initial condition:

$$\tilde{\eta}(0) \in \Omega_d \cap \Lambda\mathfrak{u}(n)_\tau \,.$$

Comparing (4.11) with (4.9) and (4.10) we conclude that $\tilde{\Phi}$ is an extended solution of finite type if

$$-\frac{i}{2}\pi\bar{\partial}\pi = \pi\eta_{-d+1}\pi^{\perp} \,. \tag{4.12}$$

Lemma 13. Equation (4.12) holds for $\alpha = \underline{\mathrm{Ker}}A'_\psi$.

Proof. Since A'_ψ is minus the adjoint of $A''_{\psi^\perp}$ and ψ is of finite type, we have

$$\underline{\psi} \ominus \underline{\alpha} = \mathrm{Im}A''_{\psi^\perp}$$

everywhere. On the other hand, equations (4.3) and (4.8) give

$$A''_{\psi^\perp} = -2i\pi_\psi\eta_{-d}\pi_{\psi^\perp} \,.$$

Take a smooth section v of $\underline{\psi}^\perp$. Then $\eta_{-d}v \in C^\infty(\underline{\psi} \ominus \underline{\alpha})$. Considering equation (4.6) together with conditions (4.5) we obtain:

$$\begin{aligned}
(\pi\bar{\partial}\pi)(\eta_{-d}v) &= -\pi\bar{\partial}(\eta_{-d}v) = -\pi(\bar{\partial}\eta_{-d})v \\
&= 2i\pi[\eta_{-d+1}, \eta_{-d}]v \\
&= 2i\pi\eta_{-d+1}\eta_{-d}v - 2i\pi\eta_{-d}\eta_{-d+1}v \\
&= 2i\pi\eta_{-d+1}(\eta_{-d}v) \,.
\end{aligned}$$

Whence

$$-\frac{i}{2}(\pi\bar{\partial}\pi)\pi_{\psi\ominus\alpha} = \pi\eta_{-d+1}\pi_{\psi\ominus\alpha} \,. \tag{4.13}$$

At same time,

$$(\pi\bar{\partial}\pi)(v) = -\pi A''_{\psi^\perp}(v) = 0 \,;$$

and since $\eta_{-d+1} \in \psi^*[\mathfrak{k}]^{\mathbb{C}}$, we also have $\pi\eta_{-d+1}\pi_{\psi^\perp} = 0$. Thus

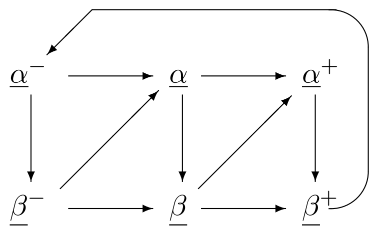
$$-\frac{i}{2}(\pi\bar{\partial}\pi)\pi_{\psi^\perp} = \pi\eta_{-d+1}\pi_{\psi^\perp} \,. \tag{4.14}$$

From equations (4.13) and (4.14) we conclude that (4.12) holds for $\alpha = \underline{\mathrm{Ker}}A'_\psi$. ☐

Corollary 1. If $\psi : \mathbb{C} \to G_k(\mathbb{C}^n)$ is of finite type, then $G^{(r)}(\psi)$ is of finite type for every integer r.

Proof. If ψ is of finite type so it is $\psi^{\perp}$. Note that $\underline{\mathrm{Ker}}A'_{\psi^{\perp}} = \psi^{\perp} \ominus G^{(-1)}(\psi)$ and $G^{(-1)}(\psi)$ is obtained by flag transform from the pair $(\psi^{\perp}, \underline{\mathrm{Ker}}A'_{\psi^{\perp}})$. Applying Lemma (13) we conclude that $G^{(-1)}(\psi)$ is of finite type. Iterating this argument and reversing orientation we have the result. □

We shall now give another example that will be useful for us: suppose we have an ∂- and $\bar{\partial}$-irreducible harmonic map $\psi : \mathbb{C} \to G_2(\mathbb{C}^6)$ of isotropy order $r = 2$ such that the first ∂-return map $c'_2(\psi)$ is nilpotent and rank $\underline{\mathrm{Im}}c'_2(\psi) = 1$. Moreover, suppose that rank $G^{(2)}(\psi) = 1$. Then, associated to ψ we have the following diagram:



where $\underline{\alpha}$ is the image of the first ∂-return map of ψ, $\underline{\beta} = \underline{\psi} \ominus \underline{\alpha}$, $\underline{\alpha}^{+} = \underline{\mathrm{Im}}(A'_{\psi} \mid \underline{\alpha})$, $\underline{\alpha}^{-} = \underline{\mathrm{Im}}A'_{G^{(1)}(\psi)}$, $\underline{\beta}^{+} = G^{(1)}(\psi) \ominus \underline{\alpha}^{+}$ and $\underline{\beta}^{-} = G^{(-1)}(\psi) \ominus \underline{\alpha}^{-}$.

Remark. Here we are using diagrams as in [2], [13] and [46]; namely, if $\underline{\psi}^1, \ldots \underline{\psi}^s$ is a collection of muttually orthogonal subbundles, then each second fundamental form A'_{ψ^i,ψ^j} will be represented by an arrow from $\underline{\psi}^i$ to $\underline{\psi}^j$; with no arrow shown when A'_{ψ^i,ψ^j} is known to vanish.

Lemma 14. If $\underline{\psi} = \underline{\alpha} \oplus \underline{\beta}$ is harmonic of finite type, then $\underline{\phi} = \underline{\alpha}^{+} \oplus \underline{\beta}$ is also harmonic of finite type.

Proof. By Corollary 1 $G^{(-1)}(\psi)$ is of finite type. Let ξ be a Killing field associated to $\underline{\varphi} = G^{(-1)}(\psi)$ and π the Hermitian projection onto $\underline{\gamma} = \underline{\alpha}^{-} \oplus \alpha$. Note that $\underline{\gamma}$ satisfies the ∂-replacement conditions with respect to φ, hence $\underline{\phi}^{-} = \underline{\beta}^{-} \oplus \alpha$ is harmonic.

Since φ is of finite type, we have

$$\underline{\alpha}^{-} = \mathrm{Im}A'_{\varphi^{\perp}}\ , \ \underline{\alpha} = \mathrm{Im}A'_{\varphi} \circ A'_{\varphi^{\perp}}$$

everywhere. Equations (4.3) and (4.8) give $2i\xi_d = A'_{\varphi} + A'_{\varphi^{\perp}}$.

Take a smooth section v of $\varphi^\perp$. Then $\xi_d v \in C^\infty(\underline{\alpha}^-)$ and $\xi_d\xi_d v \in C^\infty(\underline{\alpha})$. Considering equation (4.6) together with conditions (4.5) we obtain:

$$\begin{aligned}(\pi^\perp\partial\pi)(\xi_d v) &= \pi^\perp\partial(\xi_d)v + \pi^\perp\xi_d\partial v\\ &= 2i\pi^\perp[\xi_{d-1},\xi_d]v + \pi^\perp\xi_d\partial v\\ &= 2i\pi^\perp\xi_{d-1}\xi_d v - \underbrace{2i\pi^\perp\xi_d\xi_{d-1}v}_{=0} + \underbrace{\pi^\perp\xi_d\partial v}_{=0}\\ &= 2i\pi^\perp\xi_{d-1}(\xi_d v)\end{aligned}$$

and

$$\begin{aligned}(\pi^\perp\partial\pi)(\xi_d\xi_d v) &= \pi^\perp\partial(\xi_d\xi_d v)\\ &= \pi^\perp(\partial\xi_d)\xi_d v + \pi^\perp\xi_d(\partial\xi_d)v + \underbrace{\pi^\perp\xi_d\xi_d\partial v}_{=0}\\ &= 2i\pi^\perp[\xi_{d-1},\xi_d]\xi_d v + 2i\pi^\perp\xi_d[\xi_{d-1},\xi_d]v\\ &= 2i\pi^\perp\xi_{d-1}\xi_d\xi_d v - \underbrace{2i\pi^\perp\xi_d\xi_d\xi_{d-1}v}_{=0}\\ &= 2i\pi^\perp\xi_{d-1}\xi_d\xi_d v\,.\end{aligned}$$

Hence

$$-\frac{i}{2}\pi^\perp\partial\pi = \pi^\perp\xi_{d-1}\pi\,,$$

and conjugating this equation we get

$$-\frac{i}{2}\pi\bar\partial\pi^\perp = \pi\xi_{d-1}\pi^\perp\,,$$

which means that (4.12) holds for the pair $(\varphi,\underline{\gamma})$. So we conclude that $\underline{\phi}^-$ is of finite type. The Lemma follows from the fact that $\underline{\phi}$ is just the first Gauss bundle of $\underline{\phi}^-$. □

However, flag transforms do not always preserve finite type: let $\phi:\mathbb{C}\to G_k(\mathbb{C}^n)$ be a (non-constant) harmonic map of finite type and $\alpha:\mathbb{C}\to G_s(\mathbb{C}^n)$ a (non-constant) holomorphic map. Then $\psi=\phi\oplus\alpha:\mathbb{C}\to G_{k+s}(\mathbb{C}^n)$ is a harmonic map which is obtained by flag transform from the pair $(\phi,\underline{\alpha})$. If A'_α has singular points (points where the rank of $\mathrm{Im}A'_\alpha$ drops), the same happens to $A'_\psi = A'_\phi\oplus A'_\alpha$. Then we conclude from Lemma 12 that in this case ψ can not be of finite type. We shall see below that the second fundamental form A'_α of a non-constant doubly periodic holomorphic map $\alpha:\mathbb{C}\to\mathbb{C}P^1$ must have singular points.

4.4 Finite uniton number vs. finite type

The notions of harmonic maps of finite type and finite uniton number are fundamentally distinct. To illustrate this we start by considering the example of harmonic maps from a surface M to a complex projective space. Recall the well known characterization of harmonic maps $M \to \mathbb{C}P^n$ of finite uniton number:

Theorem 23. [21][38][44] Let $\phi : M \to \mathbb{C}P^n$ be a harmonic map. Then ϕ is of finite uniton number if and only if $\underline{\phi} = G^{(i)}(f)$ for some holomorphic (whence harmonic) map $f : M \to \mathbb{C}P^n$ and some integer i.

Otherwise said, a harmonic map $\phi : M \to \mathbb{C}P^n$ of finite uniton number is holomorphic, anti-holomorphic or is covered by a horizontal holomorphic map

$$\psi : \mathbb{C} \to \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(1) \times \mathrm{U}(n-r-1)),$$

with $r \geq 1$.

The situation for harmonic maps of finite type is completely different:

Theorem 24. Let $\phi : \mathbb{C} \to \mathbb{C}P^n$ be a doubly periodic harmonic and T the corresponding torus. If ϕ is simultaneously of finite type and of finite uniton number then ϕ is constant.

Proof. Start to suppose that $\phi : T \to \mathbb{C}P^n$ is holomorphic and of finite type. Then, the corresponding sequence of Gauss bundles must be finite:

$$\phi_{-1} = 0, \phi = \phi_0, \phi_1, \ldots, \phi_r, \phi_{r+1} = 0$$

(denoting $\phi_i = G^{(i)}(f)$). By Corollary 1 each ϕ_i is also of finite type. Hence, since the canonical line bundle over T is trivial, each $\mathcal{A}'_{\phi_i}$ induces a vector bundle isomorphism $\phi_i \to \phi_{i+1}$. Then

$$c_1(\phi_0) = c_1(\phi_1) = \ldots = c_1(\phi_r), \tag{4.15}$$

where $c_1(\ell)$ is the first Chern number of the line bundle ℓ over the corresponding torus T. By [49] we know that

$$c_1(\phi_i) = \frac{\sqrt{-1}}{2\pi} \int_T (|A'_{\phi_{i-1}}|^2 - |A'_{\phi_i}|^2)\mathrm{dz} \wedge \mathrm{d}\bar{z}. \tag{4.16}$$

It follows now from (4.15) and (4.16) that

$$c_1(\phi_0) = \frac{1}{r+1}\sum_{i=0}^{r} c_1(\phi_i) = \frac{1}{r+1}\{\frac{\sqrt{-1}}{2\pi}\int_T (|A'_{\phi_{-1}}|^2 - |A'_{\phi_r}|^2)\mathrm{d}z \wedge \mathrm{d}\bar{z}\} = 0$$

since ϕ_0 is holomorphic (whence $|A'_{\phi_{-1}}| = 0$) and ϕ_r is anti-holomorphic (whence $|A'_{\phi_r}| = 0$). So

$$0 = c_1(\phi_0) = -\frac{\sqrt{-1}}{2\pi}\int_T |A'_{\phi_0}|^2 \mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{4.17}$$

and we conclude that $A'_{\phi_0} = 0$, i.e. $\phi = \phi_0$ is constant.

Back to the general case, suppose that $\phi : T \to \mathbb{C}P^n$ is simultaneously of finite type and of finite uniton number (not necessarily holomorphic). Being of finite uniton number, by Theorem 23 there exists some holomorphic map $g : T \to \mathbb{C}P^n$ such that $\underline{\phi} = G^{(j)}(g)$ for some $j \geq 0$; this map g is simultaneously of finite type (because it is the $(-i)^{th}$-Gauss bundle of ϕ) and of finite uniton number (because it is holomorphic), hence g must be constant; so $\phi = g$ and we are done. $\square$

Remark. Let $\phi : T \to \mathbb{C}P^1$ be holomorphic. Suppose $\mathcal{A}'_{\phi_i} = \mathrm{d}z \otimes A'_\phi$ induces a vector bundle isomorphism $\underline{\phi} \to \underline{\phi}^\perp$. Then, $c_1(\underline{\phi}) = c_1(\underline{\phi}^\perp)$. At same time, by [13] (lemma 5.1) $c_1(\underline{\phi}) = -c_1(\underline{\phi}^\perp)$; thus $c_1(\underline{\phi}) = 0$. Applying (4.17) we see that in this case ϕ must be constant, which is a contradiction. So, for any holomorphic map $\phi : T \to \mathbb{C}P^1$, A'_ϕ must have singular points.

Remark. Let $\phi : T \to \mathbb{C}P^{n-1}$ be a (non-constant) harmonic map of finite type and $\alpha : T \to \mathbb{C}P^1$ a (non-constant) holomorphic map. Then the harmonic map $\psi = \phi \oplus \alpha : T \to G_2(\mathbb{C}^n)$ can not be of finite type neither of finite uniton number (if ψ was of finite uniton number, ϕ would be simultaneously of finite uniton number and finite type, whence ϕ would be constant).

Given a k-symmetric space F and a map $\psi : M \to F$, following [12] we say that ψ is *super-horizontal* if $\mathrm{d}\psi(TM^{\mathbb{C}}) \subset [\mathfrak{g}_1] \oplus [\mathfrak{g}_{-1}]$. We shall now give a generalization of Theorem 24:

Theorem 25. Let $\phi : \mathbb{C} \to G_k(\mathbb{C}^n)$ be a doubly periodic harmonic map and T the corresponding torus. Suppose that ϕ has finite uniton number. Moreover, suppose that ϕ is covered by a super-horizontal holomorphic map $\psi : M \to F_I$, where F_I is the generalized flag manifold defined by

$$\psi : \mathbb{C} \to \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(i_1) \times \ldots \times \mathrm{U}(n - i_{r-1}))\,.$$

If ϕ is of finite type, then ϕ is constant.

Proof. Since ϕ is covered by a super-horizontal holomorphic map $\psi : \mathbb{C} \to F_I$, ϕ is of the form

$$\phi = \sum_{i=1}^{s} \alpha_i \ominus \alpha_{i-1}\,,$$

with $\alpha_0 \subsetneq \alpha_1 \subsetneq \ldots \subsetneq \alpha_n \subset \mathbb{C}^n$ holomorphic subbundles, since ψ is holomorphic, and $\operatorname{Im} A'_{\alpha_i} \subset \alpha_{i+1}$, since ψ is super-horizontal (cf.[12]). This means that the corresponding Gauss sequence must be finite:

$$0 = \phi_{-r}, \phi_{-r+1}, \ldots, \phi, \ldots, \phi_{l-1}, \phi_l = 0\,, \tag{4.18}$$

where $\phi_i = G^{(i)}(\phi)$. By Corollary 1 each ϕ_i is of finite type. Moreover, each ϕ_i certainly is covered by some super-horizontal holomorphic map $\psi_i : \mathbb{C} \to F_{I_i}$. If all the bundles $\phi_{-r+1}, \ldots, \phi, \ldots, \phi_{l-1}$ have the same dimension, then

$$c_1(\phi_{-r+1}) = \ldots = c_1(\phi) = \ldots c_1(\phi_{l-1})\,,$$

where $c_1(\phi_i) = c_1(\bigwedge^k \phi)$. In this case, since formula (4.16) also holds for each $c_1(\phi_i)$, the same argument we have used to prove Theorem 24 can be applied here to conclude that ϕ must be constant.

Suppose now that the sequence (4.18) degenerates, in the sense that

$$0 < \dim\, \phi_i < \dim\, \phi$$

for some i. In this case, ϕ_i can not be constant; hence the Gauss sequence associated to ϕ_i degenerates. Iterating this procedure, we will end up with a contradiction. Then we conclude that ϕ must be constant.

□

4.5 Harmonic tori in $\mathbb{H}P^n$

It was shown in [9] that *any non-conformal harmonic map of a 2-torus into a rank one symmetric space is of finite type*. After that, Burstall [7] proved that *any weakly conformal non-isotropic harmonic map of a 2-torus into a sphere or a complex projective space can be lifted to a primitive map of finite type into a certain k-symmetric space*. Combining these results with Theorems 18 and 19 we conclude that *any harmonic map of 2-torus into a sphere or a complex projective space is of finite type or of finite uniton number*. Thus a natural question arise: does this remain true for any other rank one symmetric space? Unfortunately, as we shall show now, the answer is *no* in quaternionic projective space case.

Let $\mathbb{H}$ denote the division ring of quaternions. Let $\jmath$ be a unit quaternion with $\jmath^2 = -1$. We have an identification of $\mathbb{C}^2$ with $\mathbb{H}$ given by making

$(a, b) \in \mathbb{C}^2$ correspond to $a + b\jmath \in \mathbb{H}$; we then have a corresponding identification of $\mathbb{C}^{2n}$ with $\mathbb{H}^n$. Let $J : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the conjugate linear map given by left multiplication by $\jmath$:

$$J(z_1, z_2, \ldots, z_{2n-1}, z_{2n}) = (-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2n}, \bar{z}_{2n-1});$$

then $J^2 = -Id$. We shall regard the quaternionic projective space $\mathbb{H}P^{n-1}$ as the totally geodesic submanifold of $G_2(\mathbb{C}^{2n})$ given by

$$\mathbb{H}P^{n-1} = \{V \in G_2(\mathbb{C}^{2n}) : V = JV\}.$$

Note that the operator J satisfies

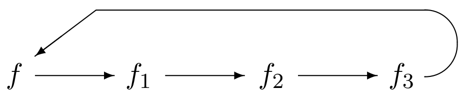
$$\langle Jv, Jw \rangle = \langle w, v \rangle \text{ for all } v, w \in \mathbb{C}^{2n},$$

in particular v is perpendicular to Jv for all $v \in \mathbb{C}^{2n}$.

Consider the smooth doubly periodic map $f : \mathbb{C} \to \mathbb{C}P^3$ (with periods $\omega_1 = \frac{\pi}{2}$, $\omega_2 = i\frac{\pi}{2}$) defined by

$$f(z, \bar{z}) = [(e^{2i(z+\bar{z})}, ie^{-2i(z+\bar{z})}, e^{2(\bar{z}-z)}, e^{2(z-\bar{z})})]\,.$$

Let Λ be the lattice generated by ω_1, ω_2. Then f descends to the torus $T = \mathbb{C}/\Lambda$ and by simple analysis of the derivatives of f we can conclude that f is full harmonic with isotropy order $r = 3$ (hence f is a harmonic map of finite type); moreover, $f_1 = Jf$ and, consequently, $f_2 = Jf_3$, where f_i denotes the i^{th}-Gauss bundle of f. Associated to f we have the following diagram:



Note that

$$v = \frac{1}{\sqrt{8}}(e^{2i(z+\bar{z})}, ie^{-2i(z+\bar{z})}, e^{2(\bar{z}-z)}, e^{2(z-\bar{z})}) \tag{4.19}$$

provides a global non-vanishing holomorphic section of f such that

$$\langle \partial v, Jv \rangle = 1\,.$$

Now we want to construct from f a harmonic map $\phi : T \to \mathbb{H}P^2$ which is not of finite type neither of finite uniton number. For that we start to state the following two lemmas:

Lemma 15. There is a pair (θ, ϑ) of non-constant meromorphic functions on $T = \mathbb{C}/\Lambda$ satisfying $\theta' = \vartheta^2$.

Proof. Consider the $\wp$-function associated to the lattice Λ:

$$\wp(z) = z^{-2} + \sum_{\omega\in\Lambda-\{0\}} [(z-\omega)^{-2} - \omega^{-2}]\,,$$

which satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3\,, \tag{4.20}$$

where

$$g_2 = 60\sum_{\omega\in\Lambda}\omega^{-4} \quad g_3 = 140\sum_{\omega\in\Lambda}\omega^{-6}\,.$$

By taking the derivative of (4.20) we deduce that

$$\wp'' = 6\wp^2 - \frac{g_2}{2}; \tag{4.21}$$

whence

$$\wp''' = 12\wp'\wp\,. \tag{4.22}$$

But $g_3 = 0$ for any square lattice (cf.[32]); then, from (4.20), (4.21) and (4.22) it follows that

$$\begin{aligned}(\wp'\wp'')' &= (\wp'')^2 + \wp'''\wp' \\ &= 84\wp^4 - 18g_2\wp^2 + \frac{g_2^2}{4}\,;\end{aligned}$$

and for $\lambda \in \mathbb{C}$:

$$\begin{aligned}(\lambda\wp' + \wp'\wp'')' &= \lambda\wp'' + (\wp'\wp'')' \\ &= \underbrace{84}_{a}\wp^4 + \underbrace{(6\lambda - 18g_2)}_{b}\wp^2 + \underbrace{\frac{g_2^2}{4} - \lambda\frac{g_2}{2}}_{c}\,.\end{aligned} \tag{4.23}$$

$\Delta = b^2 - 4ac$ is a polynomial of degree 2 in λ, which admits some root $\lambda_0 \in \mathbb{C}$. Then we conclude from (4.23) that

$$(\lambda_0\wp' + \wp'\wp'')' = (A\wp^2 + B)^2$$

for some $A, B \in \mathbb{C}$, with $A \neq 0$; which means that $\theta = \lambda_0\wp' + \wp'\wp''$ and $\vartheta = A\wp^2 + B$ are non-constant meromorphic functions on T satisfying $\theta' = \vartheta^2$. $\square$

Lemma 16. There is a holomorphic line subbundle γ of $f \oplus \underline{\mathbb{C}}^2$ such that $\langle \partial u, Ju\rangle = 0$ for every smooth section u of γ, that is, $\pi_{J\gamma}A'_\gamma = 0$.

Proof. Let (θ, ϑ) be a pair of non-constant meromorphic functions on T satisfying $\theta' = \vartheta^2$ and e_1, Je_1 an orthonormal basis of $\mathbb{C}^2$. Set $s_1 = \vartheta v$, with v given by (4.19) $s_2 = \theta e_1 + Je_1$ and $s = s_1 + s_2$. Then we have:

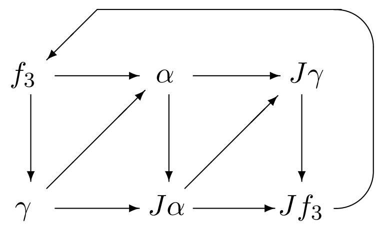
$$\begin{aligned}
\langle \partial s, Js \rangle &= \langle \partial s_1 + \partial s_2, Js_1 + Js_2 \rangle \\
&= \langle \partial s_1, Js_1 \rangle + \langle \partial s_2, Js_2 \rangle \\
&= \vartheta^2 \underbrace{\langle \partial v, Jv \rangle}_{=1} - \theta' \underbrace{\langle e_1, e_1 \rangle}_{=1} \\
&= \vartheta^2 - \theta' \\
&= 0\,.
\end{aligned}$$

Then, the line subbundle γ of $f \oplus \mathbb{C}^2$ generated by s satisfies $\pi_{J\gamma} A'_\gamma = 0$. $\square$

Let δ be the holomorphic line subbundle of $\underline{\mathbb{C}}^2$ generated by $s_2 = \theta e_1 + Je_1$. Hence δ is harmonic with Gauss bundles $G^{(1)}(\delta) = J\delta$, $G^{(2)}(\delta) = 0$. Setting $\psi = f \oplus \delta \subset \underline{\mathbb{C}}^4 \oplus \underline{\mathbb{C}}^2$ we obtain an harmonic map which is not of finite type neither of finite uniton number with Gauss bundles $G^{(i)}(\psi) = G^{(i)}(f) \oplus G^{(i)}(\delta)$. In particular,

$$\begin{cases} G^{(1)}(\psi) = Jf \oplus J\delta = J\psi \\ G^{(2)}(\psi) = Jf_3 \text{ and } G^{(3)}(\psi) = f_3. \end{cases}$$

The line bundle γ is clearly holomorphic in ψ. Let α be the orthogonal complement of γ in ψ: $\psi = \gamma \oplus \alpha$. Then we can construct the following diagram:



From this diagram we see that $\phi = \alpha \oplus J\alpha : T \to \mathbb{H}P^2$ is a harmonic map obtained by flag transform from the pair $(\psi, \gamma \oplus J\alpha)$. So, from Lemma 14 we conclude that ϕ can not be of finite type neither of finite uniton number.

4.6 Further work

1. The result we have obtained in section 4.4 suggests the following question: are there non-constant harmonic maps into $\mathrm{U}(n)$ which are simultaneously of finite uniton number and of finite type?

2. The example we have found in section 4.5 shows that the theory of harmonic 2-tori in $\mathbb{H}P^n$ is far more complicated then that of harmonic 2-tori in $\mathbb{C}P^n$ or S^n. Here we shall recall some results due to Udagawa: In [43], the author showed that any harmonic map from the 2-torus into $\mathbb{H}P^n$ of odd isotropy order is covered by a primitive map of finite type into some k-symmetric space. The same happens with the harmonic maps of even isotropy order and non-singular first return map. Thus, considering the results of Chapter 4, a considerable class of harmonic maps of finite type from the 2-torus into $\mathbb{H}P^n$ have already been detected. Moreover, for $n = 2, 3$, Udagawa showed that any harmonic map $\phi : T^2 \to \mathbb{H}P^n$ is of finite uniton number, finite type or is obtained out of some harmonic map of finite type into a Grassmannian after adding a finite number of unitons. Thus we can ask if a similar characterization is also possible for $n > 3$.

Chapter 5

Harmonic two-spheres in $\mathrm{Sp}(n)$

In this chapter, we shall make use of the Grassmannian theoretic point of view introduced by Segal [38] in order to study harmonic maps from a two-sphere into $\mathrm{Sp}(n)$. By using this methodology, we shall be able to deduce i) an "uniton factorization" of such maps (section 5.2.3) and ii) an alternative characterization of harmonic two-spheres in $\mathbb{H}P^n$ (section 5.4). Such a characterization can be seen as a natural generalization of Aithal's characterization of harmonic two-spheres in $\mathbb{H}P^2$ [1] which can be related to the approach of Bahy-El-Dien and Wood in [2].

5.1 Grassmannian model for loop groups

Fix on $\mathbb{C}^n$ the standard Hermitian metric $\langle \cdot, \cdot \rangle$. Let $H^{(n)}$ be the Hilbert space $L^2(S^1; \mathbb{C}^n)$ of square-summable $\mathbb{C}^n$-valued functions on the circle. If $e_1, \ldots, e_n$ denote the standard basis vectors for $\mathbb{C}^n$, then the functions

$$\lambda \mapsto \lambda^i e_j,$$

with $i \in \mathbb{Z}$ and $j = 1, \ldots, n$, are a basis for $H^{(n)}$. In other words,

$$H^{(n)} = \mathrm{Span}\{\lambda^i e_j : i \in \mathbb{Z},\, j = 1, \ldots, n\},$$

where "Span$\{X\}$" means "the closed subspace generated by X". Those functions with $i \geq 0$ generate a closed subspace

$$H_+^{(n)} = \mathrm{Span}\{\lambda^i e_j : i \geq 0,\, j = 1, \ldots, n\}$$

of $H^{(n)}$. Let Grass($H^{(n)}$) denote the space of all vector subspaces $W \subset H^{(n)}$ such that:

a) W is closed;

b) the projection map $W \to H_+^{(n)}$ is Fredholm, and the projection map $W \to (H_+^{(n)})^\perp$ is Hilbert-Schmidt;

c) the images of the projections maps $W^\perp \to H_+^{(n)}$, $W \to (H_+^{(n)})^\perp$ are contained in $C^\infty(S^1; \mathbb{C}^n)$.

The meaning of "Fredholm" and "Hilbert-Schmidt" can be found in books on functional analysis (see, for example, [37]); we do not use these concepts explicitly, so we omit any further explanation of them.

Definition 5. $\mathrm{Gr}^{(n)} = \{W \in \mathrm{Grass}(H^{(n)}) : \lambda W \subset W\}$

One of the basic results of Presseley and Segal [35] is:

Theorem 26. (*Grassmannian model of* $\Omega\mathrm{U}(n)$)

$$\mathrm{Gr}^{(n)} \cong \Lambda\mathrm{U}(n)/\mathrm{U}(n) \cong \Lambda\mathrm{GL}(n,\mathbb{C})/\Lambda^+\mathrm{GL}(n,\mathbb{C})$$

(and hence $\mathrm{Gr}^{(n)} \cong \Omega\mathrm{U}(n)$).

The method of proof is to show that $\Lambda\mathrm{GL}(n,\mathbb{C})$ acts on $\mathrm{Gr}^{(n)}$ in a natural way, and that the subgroup $\Lambda\mathrm{U}(n)$ acts transitively. The formula for the action is simply

$$\gamma \cdot W = \{\gamma f : f \in W\}.$$

By considering Fourier series, it is easy to see that the isotropy subgroup at $H_+^{(n)}$ is $\Lambda^+\mathrm{GL}(n,\mathbb{C})$ (for the action of $\Lambda\mathrm{GL}(n,\mathbb{C})$), or $\mathrm{U}(n)$ (for the action of the subgroup $\Lambda\mathrm{U}(n)$).

Remarks.

1. Given $W \in \mathrm{Gr}^{(n)}$, it is known that $\dim W \ominus \lambda W = n$ (cf. [35]), where $W \ominus \lambda W$ denotes the orthogonal complement of λW in W. If we choose an orthonormal basis for $W \ominus \lambda W$, $\{w_1, \dots, w_n\}$, we can put the vector-valued functions w_i side by side to form an $(n \times n)$-matrix valued function γ on S^1, that is, a loop $\gamma \in \Lambda\mathrm{U}(n)$. It can be shown that $\gamma \cdot H_+^{(n)} = W$ (cf. [35]).

2. A consequence of this theorem is that any loop $\gamma \in \Lambda\mathrm{GL}(n,\mathbb{C})$ can be factorized uniquely

$$\gamma = \gamma_u \gamma_+ ,$$

with $\gamma_u \in \Omega\mathrm{U}(n)$ and $\gamma_+ \in \Lambda^+\mathrm{GL}(n,\mathbb{C})$. In fact, the product map is a diffeomorphism (cf. [35]). Note that $\gamma \cdot H_+^{(n)} = \gamma_u \cdot H_+^{(n)}$.

If $J : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is the anti-linear map representing left multiplication by the quaternion j ($j^2 = -1$), then X in $\mathrm{U}(2n)$ belongs to $\mathrm{Sp}(n)$ if, and only if, $XJ = JX$. The complexification of $\mathrm{Sp}(n)$ is the subgroup $\mathrm{Sp}(n, \mathbb{C})$ of all elements of $\mathrm{GL}(2n, \mathbb{C})$ which preserve the skew complex-bilinear form S defined by $S(\xi, \eta) = \langle J\xi, \eta\rangle$. The Grassmannian model of $\Omega\mathrm{Sp}(n)$ is given by:

Proposition 3. [35] A subspace $W \in \mathrm{Gr}^{(2n)}$ corresponds to a loop in $\mathrm{Sp}(n)$ if, and only if, it belongs to

$$\mathrm{Gr}^{(n)}_{\mathbb{H}} = \{W \in \mathrm{Gr}^{(2n)} : JW^{\perp} = \lambda W\} .$$

5.1.1 The algebraic Grassmannian

Let G be a connected compact Lie subgroup of $\mathrm{U}(n)$. The loop group

$$\Lambda_{\mathrm{alg}} G^{\mathbb{C}} = \{\gamma \in \Lambda G^{\mathbb{C}} : \gamma(\lambda) = \sum_{i=-k}^{k} \lambda^i A_i,\ \gamma^{-1}(\lambda) = \sum_{i=-s}^{s} \lambda^i B_i\}$$

acts on

$$\mathrm{Gr}^{(n)}_{\mathrm{alg}} = \{W \in \mathrm{Gr}^{(n)} : \lambda^k H^{(n)}_{+} \subset W \subset \lambda^{-k} H^{(n)}_{+} \text{ for some } k \in \mathbb{N}\} .$$

Not surprisingly, we have the following "algebraic" version of Theorem 26:

$$\mathrm{Gr}^{(n)}_{\mathrm{alg}} \cong \Lambda_{\mathrm{alg}} \mathrm{Gl}(n, \mathbb{C})/\Lambda^{+}_{\mathrm{alg}} \mathrm{Gl}(n, \mathbb{C}) \cong \Lambda_{\mathrm{alg}} \mathrm{U}(n)/\mathrm{U}(n) \cong \Omega_{\mathrm{alg}} \mathrm{U}(n) .$$

Birkhoff and Bruhat decompositions

Fix a maximal torus T of G with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. The integer lattice $I = (2\pi)^{-1} \exp^{-1}(e) \cap \mathfrak{t}$ may be identified with the group of homomorphisms $S^1 \to T$, by associating to $\xi \in I$ the homomorphism γ_ξ defined by $\gamma_\xi(\lambda = e^{\sqrt{-1}t}) = \exp(t\xi)$. Let us choose a fundamental Weyl chamber in $\mathfrak{t}$. The intersection of I with this will be denoted I'. I' parametrizes conjungacy classes of homomorphisms $S^1 \to G$. For each $\xi \in I'$ we write

$$\Omega_\xi = \{g\gamma_\xi g^{-1} : g \in G\} .$$

This is a complex homogeneous space. Actually, it is a flag manifold if the connected compact Lie group G is semisimple (cf. [10]).

Theorem 27. [35]

i) *Birkhoff decomposition*: $\Omega G \cdot H^{(n)}_{+} = \bigcup_{\xi \in I'} \Lambda^{-} G^{\mathbb{C}} \gamma_\xi \cdot H^{(n)}_{+}$.

ii) *Bruhat decomposition*: $\Omega G_{\mathrm{alg}} \cdot H_+^{(n)} = \bigcup_{\xi \in I'} \Lambda_{\mathrm{alg}}^+ G^{\mathbb{C}} \gamma_\xi \cdot H_+^{(n)}$.

These decompositions admit a Morse theoretic interpretation. We shall review this, following [34], [35]:

Consider the *energy* functional on paths $E : \Omega G \to \mathbb{R}$ defined by $E(\gamma) = \int_{S^1} \|\gamma'\|^2$. It is well known that the critical points of this functional are the homomorphisms $S^1 \to G$ and the (connected) critical manifolds are the conjugacy classes of such homomorphisms.

The loop group ΩG carries a structure of Kähler manifold; take the gradient with respect to the Kähler metric and write:

$$\begin{aligned} S_\xi &= \{\gamma \in \Omega G : \gamma \text{ flows into } \Omega_\xi\} \\ U_\xi &= \{\gamma \in \Omega G : \gamma \text{ flows out of } \Omega_\xi\} \end{aligned}$$

for the stable and unstable manifold of Ω_ξ, where "flow" refers to the flow of $-\nabla E$.

Let $\mathbb{C}^*_{\geq 1} = \{\lambda \in \mathbb{C} : 1 < |\lambda| < \infty\}$. This semigroup acts on ΩG by

$$(u \cdot \gamma) \cdot H_+^{(n)} = \gamma(u\lambda) \cdot H_+^{(n)} .$$

It turns out that the flow line of $-\nabla E$ starting at a point $\gamma \in \Omega G$ is given by the action of the subsemigroup $[1, \infty)$ of $\mathbb{C}^*_{\geq 1}$ on ΩG. Suppose $\gamma \cdot H_+^{(n)} \in \Lambda^- G^{\mathbb{C}} \gamma_\xi \cdot H_+^{(n)}$. Then, as $|u| \to \infty$, with $u \in \mathbb{C}^*_{\geq 1}$, we have

$$(u \cdot \gamma) \cdot H_+^{(n)} = \sum_{i \leq 0} (u\lambda)^i A_i \gamma_\xi(\lambda) \cdot H_+^{(n)} \to A_0 \gamma_\xi \cdot H_+^{(n)} ,$$

so we see explicitly that γ flows into Ω_ξ, that is, $\gamma \in S_\xi$.

The flow line of ∇E starting at γ is defined for all time if and only if $\gamma \in \Omega_{\mathrm{alg}} G$, in which case it is given by the action of the subsemigroup $(0, 1]$ of $\mathbb{C}^* = \{\lambda \in \mathbb{C} : 0 < |\lambda| < \infty\}$. Note that the formula for the action of $\mathbb{C}^*_{\geq 1}$ on ΩG extends to an action of $\mathbb{C}^*$ on $\Omega_{\mathrm{alg}} G$. Suppose $\gamma \cdot H_+^{(n)} \in \Lambda_{\mathrm{alg}}^+ G^{\mathbb{C}} \gamma_\xi \cdot H_+^{(n)}$. Then, as $|u| \to 0$, with $u \in \mathbb{C}^*$, we have

$$(u \cdot \gamma) \cdot H_+^{(n)} = \sum_{i \geq 0} (u\lambda)^i A_i \gamma_\xi(\lambda) \cdot H_+^{(n)} \to A_0 \gamma_\xi \cdot H_+^{(n)} ,$$

so we see explicitly that γ flows out of Ω_ξ, that is, $\gamma \in U_\xi$. Actually, the converse is also true:

Theorem 28. [34] The stable and unstable manifolds of Ω_ξ are given by

$$\begin{aligned} S_\xi \cdot H_+^{(n)} &= \Lambda^- G^{\mathbb{C}} \gamma_\xi \cdot H_+^{(n)} \\ U_\xi \cdot H_+^{(n)} &= \Lambda_{\mathrm{alg}}^+ G^{\mathbb{C}} \gamma_\xi \cdot H_+^{(n)} . \end{aligned}$$

The gradient flow of the energy functional E induces a map $u_\xi : U_\xi \to \Omega_\xi$ given by

$$u_\xi(\sum_{i\geq 0} \lambda^i A_i \gamma_\xi \cdot H_+^{(n)}) = A_0 \gamma_\xi \cdot H_+^{(n)} .$$

Suppose that $\lambda^k H_+^{(n)} \subset V \subset \lambda^{-k} H_+^{(n)}$. Define $p_l : H^{(n)} \to \mathbb{C}^n$ by $p_l(\sum \lambda^i a_i) = a_l$ and set $V_l = p_l(V \cap \lambda^l H_+^{(n)})$. Then it is easy to check that

$$u_\xi(V) = \lambda^{-k} V_{-k} + \ldots + \lambda^{k-1} V_{k-1} + \lambda^k H_+^{(n)} .$$

Actually, $u_\xi : U_\xi \to \Omega_\xi$ has the structure of a holomorphic vector bundle. The rank of this bundle is the Morse index of γ_ξ.

Remark. In Section 5.2, we shall use this Grassmannian point of view in order to analyze the action of the gradient flow of the Energy funcional $E : \Omega G \to \mathbb{R}$ on an extended solution $\Phi : \mathbb{C} \to \Omega G$. In fact, we shall see that if Φ is of finite uniton number then its image must lie inside one of the Bruhat cells U_ξ, except possibly for a discrete set of points $D \subset \mathbb{C}$; moreover, $u_\xi \circ \Phi : \mathbb{C} \setminus D \to \Omega_\xi$ will be a new extended solution. More details can be found in [23].

5.1.2 A factorization theorem for loops in $\Omega\mathrm{Sp}(n)$

The loop group $\Omega_{\mathrm{alg}}\mathrm{Sp}(n)$ can be identified with

$$\mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}} = \mathrm{Gr}^{(n)}_{\mathbb{H}} \cap \mathrm{Gr}^{(2n)}_{\mathrm{alg}} .$$

Define

$$\Omega^N_{\mathrm{alg}}\mathrm{Sp}(n) = \{\gamma \in \Omega_{\mathrm{alg}}\mathrm{Sp}(n) :\ \gamma = \sum_{i=-N}^{N} \lambda^i A_i\}.$$

From now on we write $H_+ = H_+^{(2n)}$. Note that $\gamma \in \Omega^N_{\mathrm{alg}}\mathrm{Sp}(n)$ if and only if the corresponding $W_\gamma \in \mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}}$ satisfies

$$W_\gamma \subset \lambda^{-N} H_+.$$

Theorem 29. Any $\gamma \in \Omega^N_{\mathrm{alg}}\mathrm{Sp}(n)$ has a factorization of the form

$$\gamma = \gamma_1 \ldots \gamma_N,$$

with each γ_i belonging to $\Omega^1_{\mathrm{alg}}\mathrm{Sp}(n)$.

Proof. If $\alpha_1, \alpha_2 \in \Omega_{\mathrm{alg}}\mathrm{Sp}(n)$ correspond to $W_1, W_2 \in \mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}}$, then $\alpha_2 = \alpha_1\beta$ for some $\beta \in \Omega^1_{\mathrm{alg}}\mathrm{Sp}(n)$ if, and only if, $W_2 \subset \lambda^{-1}W_1$. In fact, if $\alpha_2 = \alpha_1\beta$ and $W_\beta \in \mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}}$ corresponds to β, then

$$W_2 = \alpha_2 H_+ = \alpha_1\beta H_+ = \alpha_1 W_\beta\,;$$

since $W_\beta \subset \lambda^{-1}H_+$, we have

$$W_2 = \alpha_1 W_\beta \subset \lambda^{-1}\alpha_1 H_+ = \lambda^{-1}W_1\,.$$

Conversely, if $W_2 \subset \lambda^{-1}W_1$ then $\alpha_2 H_+ \subset \lambda^{-1}\alpha_1 H_+$ and

$${\alpha_1}^{-1}\alpha_2 H_+ \subset \lambda^{-1}H_+;$$

setting $\beta = \alpha_1^{-1}\alpha_2$ we see that $\beta \in \Omega^1_{\mathrm{alg}}\mathrm{Sp}(n)$ and $\alpha_2 = \alpha_1\beta$.

Let $\gamma \in \Omega^N_{\mathrm{alg}}\mathrm{Sp}(n)$ and W the corresponding element in $\mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}}$. Define for each $i \in \{0,\dots,N\}$ the space

$$W^i = (W \cap \lambda^{-i}H_+) + \lambda^i H_+\,. \tag{5.1}$$

We shall prove now that $W^i \in \mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}}$:

i) Since $\lambda W \subset W$, we have

$$\lambda W^i = (\lambda W \cap \lambda^{-i+1}H_+) + \lambda^{i+1}H_+ \subset (W \cap \lambda^{-i}H_+) + \lambda^i H_+ = W^i\,.$$

ii) Clearly W^i is algebraic:

$$\lambda^i H_+ \subset W^i \subset \lambda^{-i}H_+\,.$$

iii) Note that each W^i can be regarded as an element of the Hermitian finite-dimensional J-stable vector space

$$E^N = \lambda^{-N}H^{(n)}_+/\lambda^N H^{(n)}_+ \cong \sum_{i=-N}^{N-1} \lambda^i\mathbb{C}^n\,,$$

and we know that for a general finite-dimensional Hermitian vector space V the following relations hold:

$$\begin{cases} (A+B)^\perp = A^\perp \cap B^\perp \\ (A\cap B)^\perp = A^\perp + B^\perp \\ (A+B)\cap C = A + (B\cap C) \text{ if } A \subset C \end{cases} \tag{5.2}$$

where A, B and C are subspaces of V. Using these we obtain:

$$\begin{aligned}
(JW^i)^\perp &= \{(JW \cap \lambda^i JH_+) + \lambda^{-i} JH_+\}^\perp \\
&= \{JW \cap \lambda^i JH_+\}^\perp \cap \{\lambda^{-i} JH_+\}^\perp \\
&= \{JW^\perp + \lambda^i JH_+{}^\perp\} \cap \{\lambda^{-i} JH_+{}^\perp\} \\
&= \{\lambda W + \lambda^{i+1} H_+\} \cap \{\lambda^{-i+1} H_+\} \\
&= (\lambda W \cap \lambda^{-i+1} H_+) + \lambda^{i+1} H_+ = \lambda W^i\,.
\end{aligned}$$

So $W^i \in \mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}}$.

On the other hand,

$$\begin{aligned}
W^{i+1} &= (W \cap \lambda^{-i-1} H_+) + \lambda^{i+1} H_+ \\
&= \lambda^{-1}\{(\lambda W \cap \lambda^{-i} H_+) + \lambda^{i+2} H_+\} \\
&\subset \lambda^{-1}\{(W \cap \lambda^{-i} H_+) + \lambda^{i} H_+\} = \lambda^{-1} W^i\,.
\end{aligned}$$

Hence $W^{i+1} \subset \lambda^{-1} W^i$ and the sequence

$$H_+ = W^0, W^1, \ldots, W^N = \gamma H_+$$

defines a factorization of γ:

$$\gamma = \gamma_1 \gamma_2 \ldots \gamma_N\,,$$

with $\gamma_i \in \Omega^1_{\mathrm{alg}}\mathrm{Sp}(n)$. □

There exists a canonical refinement of such a factorization into "linear factors":

For any subspace V of $\mathbb{C}^{2n}$, let γ_V denote the element of $\Omega\mathrm{U}(2n)$ given by $\gamma_V(\lambda) = \pi_V + \lambda \pi_{V^\perp}$. We say that V is *J-isotropic* if $V \subset JV^\perp$. The elements γ_V of $\Omega\mathrm{U}(2n)$ correspond precisely to the subspaces $W \in \mathrm{Gr}^{(2n)}$ such that $\lambda H_+ \subset W \subset H_+$. More generally, if $\alpha_1, \alpha_2 \in \Omega\mathrm{U}(2n)$ correspond to $W_1, W_2 \in \mathrm{Gr}^{(2n)}$, then $\alpha_2 = \alpha_1 \gamma_V$ for some subspace $V \in \mathbb{C}^{2n}$ if and only if $\lambda W_1 \subset W_2 \subset W_1$.

Let $\gamma \in \Omega^N_{\mathrm{alg}}\mathrm{Sp}(n)$ and W the corresponding element of $\mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}}$. Consider the canonical factorization given by Theorem 29, $\gamma = \gamma_1\gamma_2\ldots\gamma_N$. Set $F_i = \gamma_1\gamma_2\ldots\gamma_i$. Hence $W^i = F_i \cdot H_+$, where W^i is the element of $\mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}}$ given by formula (5.1). For each $i \in \{0, \ldots, N-1\}$ we define

$$W^{i^+} = \lambda W \cap \lambda^{-i} H_+ + \lambda^{i+1} H_+ \in \mathrm{Gr}^{(2n)}$$

and let F_{i^+} be the corresponding element of $\Omega\mathrm{U}(2n)$, $W^{i^+} = F_{i^+}H_+$. One can easily check that

$$\begin{array}{l} \lambda W^i \subset W^{i^+} \subset W^i \\ \lambda W^{i^+} \subset \lambda W^{i+1} \subset W^{i^+}\,; \end{array} \tag{5.3}$$

then $F_{i^+} = F_i \gamma_{V_i}$ and $F_{i+1} = F_{i^+}{\gamma_{\tilde{V}_i}}^{-1}$ for some vector subspaces $V_i, \tilde{V}_i$ of $\mathbb{C}^{2n}$. The sequence

$$H_+ = W^0, W^{0^+}, W^1, W^{1^+}, \dots, W^N = \gamma H_+$$

induces a factorization of γ into "linear factors"

$$\gamma = \underbrace{\gamma_{V_0}{\gamma_{\tilde{V}_0}}^{-1}}_{\gamma_1} \underbrace{\gamma_{V_1}{\gamma_{\tilde{V}_1}}^{-1}}_{\gamma_2} \cdots \underbrace{\gamma_{V_{N-1}}{\gamma_{\tilde{V}_{N-1}}}^{-1}}_{\gamma_N}\,.$$

These factors are not in $\Omega\mathrm{Sp}(n)$. However, they have some J-structure. In fact, it turns out that the subspaces $V_i, \tilde{V}_i$ are J-isotropic:

Note that $\gamma_{V_i} \cdot H_+ = V_i + \lambda H_+$ and $J\gamma_{V_i} \cdot H_+ = JV_i + {H_+}^{\perp}$. Hence V_i is isotropic if and only if $J\gamma_{V_i} \cdot H_+ \subset (\gamma_{V_i} \cdot H_+)^{\perp}$. Since $F_{i^+} = F_i\gamma_{V_i}$, we have

$$\gamma_{V_i} \cdot H_+ = {F_i}^{-1}\{(\lambda W \cap \lambda^{-i} H_+) + \lambda^{i+1} H_+\}\,;$$

then

$$J\gamma_{V_i} \cdot H_+ = {F_i}^{-1}\{(W^{\perp} \cap \lambda^{i+1} {H_+}^{\perp}) + \lambda^{-i} {H_+}^{\perp}\}\,, \tag{5.4}$$

and using equations (5.2) we obtain

$$\begin{aligned} (\gamma_{V_i} \cdot H_+)^{\perp} &= {F_i}^{-1}\{(\lambda W \cap \lambda^{-i} H_+)^{\perp} \cap \lambda^{i+1} {H_+}^{\perp}\} \\ &= {F_i}^{-1}\{(\lambda W^{\perp} + \lambda^{-i} {H_+}^{\perp}) \cap \lambda^{i+1} {H_+}^{\perp}\} \\ &= {F_i}^{-1}\{(\lambda W^{\perp} \cap \lambda^{i+1} {H_+}^{\perp}) + \lambda^{-i} {H_+}^{\perp}\}\,. \end{aligned} \tag{5.5}$$

Comparing (5.4) with (5.5) we conclude that $J\gamma_{V_i} \cdot H_+ \subset (\gamma_{V_i} \cdot H_+)^{\perp}$, that is, V_i is J-isotropic. Similarly, one can prove that each $\tilde{V}_i$ is J-isotropic.

5.2 Harmonic maps into $\mathrm{Sp}(n)$

5.2.1 Harmonic maps into Lie groups from the Grassmannian point of view

Let M be a Riemann surface. According to Segal [38], the harmonic equations for an extended solution $\Phi : M \to \Omega\mathrm{U}(n)$ associated to a harmonic map $\phi : M \to \mathrm{U}(n)$ can be reformulated in terms of $\mathrm{Gr}^{(n)}$ in the following way:

Let $W : M \to \mathrm{Gr}^{(n)}$ correspond to Φ under the identification $\Omega U_n \cong \mathrm{Gr}^{(n)}$. Thus $W(z) = \Phi(z) \cdot H_+^{(n)}$. Consider the following conditions:

$$W_z \subset \frac{1}{\lambda} W \tag{5.6}$$

$$W_{\bar{z}} \subset W\,. \tag{5.7}$$

The first condition means that $\frac{\partial s}{\partial z}(z)$ is contained in the subspace $\lambda^{-1}W(z)$ of $H^{(n)}$, for every (smooth) map $s : M \to H^{(n)}$ such that $s(z) \in W(z)$. The second condition is interpreted in a similar way; it is equivalent to saying that W is holomorphic.

Proposition 4. [38] Φ is an extended solution if and only if W is a solution of equations (5.6) and (5.7).

Definition 6. We say that $W : M \to \mathrm{Gr}^{(n)}$ is a *complex extended solution* if it is a solution of equations (5.6) and (5.7).

If M is compact and $\Phi : M \to \Omega\mathrm{U}(n)$ is an extended solution, we know, from [38] or [44], that there is a constant loop $\gamma \in \Omega\mathrm{U}(n)$ such that $\gamma\Phi$ is polynomial (in λ and λ^{-1}), that is, $\gamma\Phi : M \to \Omega_{\mathrm{alg}}\mathrm{U}(n)$. This is also true for any Lie subgroup G of $\mathrm{U}(n)$:

Proposition 5. If the Riemann surface M is compact, and $\Phi : M \to \Omega G$ is an extended solution, then there exists some $\gamma \in \Omega G$ such that $\gamma\Phi : M \to \Omega_{\mathrm{alg}}G$.

Proof. Suppose $\Phi : M \to \Omega G \subset \Omega\mathrm{U}(n)$ is an extended solution and take a constant loop γ such that $\gamma\Phi$ is polynomial. Fix some point $z_0 \in M$. Then, since $\gamma\Phi(z_0)$ is polynomial,

$$(\gamma\Phi(z_0))^{-1}\gamma\Phi = \Phi(z_0)^{-1}\Phi$$

is a loop in ΩG and is polynomial. □

Let G be a Lie subgroup of $\mathrm{U}(n)$ and $\Phi : M \to \Omega_{\mathrm{alg}}G$ an extended solution. Then, for some positive integer k, we have

$$\lambda^k H_+^{(n)} \subset W(z) \subset \lambda^{-k} H_+^{(n)}$$

for all $z \in M$, where $W : M \to \Omega_{\mathrm{alg}}G \cdot H_+$ is the complex extended solution corresponding to Φ, which means that W is a holomorphic map to a finite-dimensional complex Grassmannian manifold. Although this shows that the original Hilbert space $H^{(n)}$ will play no essential role, we continue to use it for notational convenience; the reader should bear in mind that we always work in the finite-dimensional vector space

$$E^k = \lambda^{-k} H_+^{(n)} / \lambda^k H_+^{(n)} \cong \sum_{i=-k}^{k-1} \lambda^i \mathbb{C}^n \,. \tag{5.8}$$

In particular, we can interpret W as a holomorphic vector subbundle of the trivial bundle $\underline{E}^k = M \times E^k$.

Since W is holomorphic, its image must lies inside one of the complex submanifolds

$$U_\xi \cdot H_+ = \Lambda^+_{\text{alg}} G^{\mathbb{C}} \gamma_\xi \cdot H_+^{(n)} \,,$$

except possibly for a discrete set of points $D \subset M$. We shall now analyse the action of the gradient flow of the energy functional $E : \Omega G \to \mathbb{R}$ on W:

Let k be the least integer such that

$$\lambda^k H_+ \subset W(z) \subset \lambda^{-k} H_+$$

for all $z \in M$. For each $l \in \{-k, \dots, k\}$, define the holomorphic vector bundle morphism $p_l : \underline{E}^k \to \underline{\mathbb{C}}^{2n}$ by

$$p_l(\sum a_i \lambda^i) = a_l \,. \tag{5.9}$$

Set

$$W_l = p_l(W \cap \lambda^l H_+) \,. \tag{5.10}$$

Then $u_\xi \circ W : M \setminus D \to \Omega_\xi \cdot H_+$ is given by

$$u_\xi \circ W = \lambda^{-k} W_{-k} + \dots \lambda^{k-1} W_{k-1} + \lambda^k H_+ \,.$$

Proposition 6. $\tilde{W} = u_\xi \circ W : M \setminus D \to \Omega_\xi \cdot H_+$ is a complex extended solution.

Proof. Certainly, $\tilde{W} = u_\xi \circ W$ is holomorphic, that is, $\tilde{W}_{\bar{z}} \subset \tilde{W}$ holds. On the other hand,

$$(W_i)_z = p_i(W_z \cap \lambda^i H_+) \subset p_i(\lambda^{-1} W \cap \lambda^i H_+) = p_{i+1}(W \cap \lambda^{i+1} H_+) = W_{i+1} \,,$$

which means that $\tilde{W}_z \subset \lambda^{-1} \tilde{W}$. Then $\tilde{W}$ is an extended solution. □

Remark. In [10], Burstall and Guest deduce this same result by making use of a explicit formula for $Du_\xi : T^{(1,0)} U_\xi \to T^{(1,0)} \Omega_\xi$.

S^1-invariant extended solutions

Definition 7. [10] An S^1*-invariant extended solution* is an extended solution $\Phi : M \to \Omega G$ such that $\operatorname{Im}\Phi \subset \Omega_\xi$, for some $\xi \in I'$.

Let $\Phi : M \to \Omega_\xi$ be an (S^1-invariant) extended solution . In this case, the harmonic map $\phi = \Phi_{-1} : M \to G$ factors through

$$N_\xi = \{g\gamma_\xi(-1)g^{-1} : \ g \in G\} .$$

This is a symmetric space diffeomorphic to $G/C(\gamma_\xi(-1))$, where $C(g)$ denotes the centralizer of g. The inclusion of N_ξ in G is known to be totally geodesic, with respect to the natural Riemann metrics on N_ξ and G constructed from a bi-invariant inner product on $\mathfrak{g}$. Hence, ϕ is a harmonic map to N_ξ. Observe that $\phi = \pi_\xi \circ \Phi$, where the map $\pi_\xi : \Omega_\xi \to N_\xi$ is given by $g\gamma_\xi g^{-1} \mapsto g\gamma_\xi(-1)g^{-1}$.

Suppose that the compact connected Lie group G is semisimple and has trivial centre. Let $\alpha_1, \ldots, \alpha_l$ be the simple roots associated to the fixed fundamental Weyl chamber. In this case, if $\xi \in I'$ is *canonical*, that is, if each simple root α_i takes the value 0 or $\sqrt{-1}$ on ξ, then $\pi_\xi : \Omega_\xi \to N_\xi$ is a canonical twistor fibration (cf. [10]).

5.2.2 Normalized extended solutions

Let $W : M \to \mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}}$ be a complex extended solution. The filtration of W by $W \cap \lambda^i H_+$

$$W = W \cap \lambda^{-k} H_+ \supseteq \ldots \supseteq W \cap \lambda^{k-1} H_+ \supseteq W \cap \lambda^k H_+ = \lambda^k H_+$$

induces a splitting

$$W \ominus \lambda W = A_{-k} \oplus \ldots \oplus A_k \,, \tag{5.11}$$

where

$$A_i \cong (W \cap \lambda^i H_+)/((\lambda W \cap \lambda^i H_+) + (W \cap \lambda^{i+1} H_+)) \cong W_i/W_{i-1} \tag{5.12}$$

(each W_i is given by formula (5.10)). Off a discrete set of points D, each A_i has constant dimension. Fix an integer $i > 0$ such that $A_i(z) = 0$ for all $z \in M \setminus D$. Then $W_i = W_{i-1}$ in $M \setminus D$. But $W_{-i} = J W_{i-1}{}^{\perp}$, since $\tilde{W} = u_\xi \circ W$ has values in $\mathrm{Gr}^{(n)}_{\mathbb{H},\mathrm{alg}}$; hence we also have $W_{-i} = W_{-i-1}$, that is, $A_{-i}(z) = 0$ for all $z \in M \setminus D$. Set

$$V = \lambda^i (W \cap \lambda^{-i-1} H_+) + \lambda^{1-i} (W \cap \lambda^{i-1} H_+) + \lambda H_+ \,, \tag{5.13}$$

which is clearly a well defined holomorphic map from M to $\mathrm{Gr}^{(2n)}$ except possibly for a discrete set of points.

Lemma 17. *V is a constant map into $\mathrm{Gr}^{(n)}_{\mathbb{H}}$.*

Proof. Using relations (5.2) we have:

$$\begin{aligned}
JV^{\perp} &= \{\lambda^{-i}(JW \cap \lambda^{i+1}JH_+) + \lambda^{i-1}(JW \cap \lambda^{1-i}JH_+) + \lambda^{-1}JH_+\}^{\perp} \\
&= \{(\lambda^{-i+1}W^{\perp} \cap \lambda^2 H_+^{\perp}) + (\lambda^{i}W^{\perp} \cap \lambda H_+^{\perp}) + H_+^{\perp}\}^{\perp} \\
&= \{\lambda^{-i+1}W^{\perp} \cap \lambda^2 H_+^{\perp}\}^{\perp} \cap \{\lambda^{i}W^{\perp} \cap \lambda H_+^{\perp}\}^{\perp} \cap \{H_+^{\perp}\}^{\perp} \\
&= \{\lambda^{-i+1}W + \lambda^2 H_+\} \cap \{\lambda^{i}W + \lambda H_+\} \cap H_+ \\
&= \{\lambda^{-i+1}W + \lambda^2 H_+\} \cap \{\lambda^{i}W \cap H_+ + \lambda H_+\} \\
&= \lambda^2 H_+ + \lambda^{-i+1}W \cap \{\lambda^{i}W \cap H_+ + \lambda H_+\} \\
&= \lambda^2 H_+ + \lambda^{i}W \cap H_+ + \lambda^{-i+1}W \cap \lambda H_+ \\
&= \lambda^2 H_+ + \lambda^{i}(W \cap \lambda^{-i}H_+) + \lambda^{-i+1}(W \cap \lambda^{i}H_+)\,.
\end{aligned}$$

Since $A_i = 0$ and $A_{-i} = 0$, from (5.12) we have:

$$\begin{aligned}
\lambda^{i}(W \cap \lambda^{-i}H_+) &= \lambda^{i}(\lambda W \cap \lambda^{-i}H_+) + \lambda^{i}(W \cap \lambda^{-i+1}H_+) \\
\lambda^{-i+1}(W \cap \lambda^{i}H_+) &= \lambda^{-i+1}(\lambda W \cap \lambda^{i}H_+) + \lambda^{-i+1}(W \cap \lambda^{i+1}H_+)\,.
\end{aligned}$$

But

$$\lambda^{-i+1}(W \cap \lambda^{i+1}H_+) \subset \lambda^2 H_+\,,$$

and

$$\lambda^{i}(W \cap \lambda^{-i+1}H_+) = (\lambda^{i}W \cap \lambda H_+) \subset \lambda^{-i+2}W \cap \lambda H_+ = \lambda^{-i+1}(\lambda W \cap \lambda^{i}H_+)\,.$$

So

$$JV^{\perp} = \lambda^2 H_+ + \lambda^{i}(\lambda W \cap \lambda^{-i}H_+) + \lambda^{-i+1}(\lambda W \cap \lambda^{i}H_+) = \lambda V\,,$$

that is, $V \in \mathrm{Gr}^{(n)}_{\mathbb{H}}$.

We prove now that V is constant. Since W is a complex extended solution, it satisfies equations (5.6) and (5.7). From (5.7) we see immediately that

$$V_{\bar z} \subset V\,,$$

that is, V is holomorphic. From (5.6) we have:

$$\begin{aligned}
V_z &= \lambda^{i}(W_z \cap \lambda^{-i-1}H_+) + \lambda^{1-i}(W_z \cap \lambda^{i-1}H_+) + \lambda H_+ \\
&\subset \lambda^{i}(\lambda^{-1}W \cap \lambda^{-i-1}H_+) + \lambda^{1-i}(\lambda^{-1}W \cap \lambda^{i-1}H_+) + \lambda H_+ \\
&= \lambda^{i-1}(W \cap \lambda^{-i}H_+) + \lambda^{-i}(W \cap \lambda^{i}H_+) + \lambda H_+.
\end{aligned}$$

Since $A_i = 0$ and $A_{-i} = 0$, from (5.12) we have:

$$\begin{aligned}\lambda^{i-1}(W \cap \lambda^{-i}H_+) &= \lambda^{i-1}(\lambda W \cap \lambda^{-i}H_+) + \lambda^{i-1}(W \cap \lambda^{-i+1}H_+) \\ &= \lambda^{i}(W \cap \lambda^{-i-1}H_+) + \lambda^{i-1}(W \cap \lambda^{-i+1}H_+)\end{aligned}$$

$$\begin{aligned}\lambda^{-i}(W \cap \lambda^{i}H_+) &= \lambda^{-i}(\lambda W \cap \lambda^{i}H_+) + \lambda^{-i}(W \cap \lambda^{i+1}H_+) \\ &= \lambda^{1-i}(W \cap \lambda^{i-1}H_+) + \lambda^{-i}(W \cap \lambda^{i+1}H_+)\,.\end{aligned}$$

But

$$\lambda^{-i}(W \cap \lambda^{i+1}H_+) \subset \lambda H_+$$

and

$$\lambda^{i-1}(W \cap \lambda^{-i+1}H_+) = \lambda^{i-1}W \cap H_+ \subset \lambda^{1-i}W \cap H_+ = \lambda^{1-i}(W \cap \lambda^{i-1}H_+)\,.$$

So

$$V_z \subset V\,,$$

that is, V is also anti-holomorphic, hence constant. □

Note that

$$\lambda^{k-1}V \subset W \subset \lambda^{1-k}V\,.$$

Thus, if $W = \Phi \cdot H_+$ and $V = \gamma \cdot H_+$ ($\gamma \in \Omega\mathrm{Sp}(n)$ a constant loop), $\gamma^{-1}\Phi \cdot H_+$ is a new complex extended solution, corresponding to the same original harmonic map (up to multiplication by a constant), such that

$$\lambda^{k-1}H_+ \subset \gamma^{-1}\Phi \cdot H_+ \subset \lambda^{1-k}H_+\,.$$

Since $\dim W \ominus \lambda W = 2n$, iterating this process we conclude:

Theorem 30. Let M be a compact Riemann surface and $\phi : M \to \mathrm{Sp}(n)$ a harmonic map. Suppose that there is an extended solution associated to ϕ. Then ϕ admits a complex extended solution $W : M \to \mathrm{Gr}^{(n)}_{\mathbb{H}}$ such that

$$\lambda^n H_+ \subset W \subset \lambda^{-n}H_+\,,$$

and, in the splitting (5.11), $A_i(z) \neq 0$ except perhaps at a finite set of points if $i \neq 0$. In this case, we say that W is a *normalized complex extended solution*.

Remark. Burstall and Guest prove in [10] the following result: *Assume that G is semisimple, with trivial centre. Let $\Phi : M \to \Omega^k_{\mathrm{alg}} G$ be an extended solution. Then there exists some canonical $\xi \in I'$, some $\gamma \in \Omega_{\mathrm{alg}}\mathrm{Sp}(n)$, and some discrete subset D of M such that $\gamma\Phi(M \setminus D) \subset U_\xi$.* This reduction is analogous to our concept of "normalization", which is essentially the same "'normalization" introduced by Uhlenbeck [44] and Segal [38] for the case $G = \mathrm{U}(n)$. We have proved Theorem 30 using the Grassmannian model of $\Omega\mathrm{Sp}(n)$, whereas the proof of Burstall and Guest uses the loop group ΩG directly.

5.2.3 A factorization theorem for harmonic maps into Sp(n)

In [44], Uhlenbeck proved that any harmonic map from S^2 to $\mathrm{U}(n)$ admits an extended solution $\Phi : S^2 \to \Omega_{\mathrm{alg}}\mathrm{U}(n)$ of the form

$$\Phi(\lambda) = (\pi_{f_1} + \lambda\pi^\perp_{f_1}) \dots (\pi_{f_r} + \lambda\pi^\perp_{f_r}) \, ,$$

where $f_i : S^2 \to \mathrm{Gr}_{k_i}(\mathbb{C}^n)$ for some k_i. This means that any harmonic map $\phi : S^2 \to \mathrm{U}(n)$ can be constructed from a constant map by applying a finite number of flag transforms. This was subsequently generalized by Burstall and Rawnsley [12] to the case of a group G of "type H", i.e., a compact simple Lie group which admits a Hermitian space as quotient. Segal [38] provided a new prove of Uhlenbeck's result by applying the Grassmannian point of view. We shall now follow the Segal methodology in order to give a factorization theorem for harmonic two-spheres in $G = \mathrm{Sp}(n)$.

Let M be a Riemann surface and $\phi : M \to \mathrm{Sp}(n)$ a harmonic map admitting an extended solution $\Phi : M \to \Omega^N_{\mathrm{alg}}\mathrm{Sp}(n)$. Let $W : M \to \mathrm{Gr}^{(n)}_{\mathbb{H}}$ correspond to Φ under the identification $\Omega\mathrm{Sp}(n) \cong \mathrm{Gr}^{(n)}_{\mathbb{H}}$, $W = \Phi \cdot H_+$. By Theorem 29 we know that Φ admits a factorization

$$\Phi = \tilde{\Phi}_1\tilde{\Phi}_2 \dots \tilde{\Phi}_N \, ,$$

where each $\tilde{\Phi}_i$ is a map from M to $\Omega^1_{\mathrm{alg}}\mathrm{Sp}(n)$ and

$$W^i = \tilde{\Phi}_1\tilde{\Phi}_2 \dots \tilde{\Phi}_i \cdot H_+ : M \to \mathrm{Gr}^{(n)}_{\mathbb{H}}$$

is given by

$$W^i = (W \cap \lambda^{-i} H_+) + \lambda^i H_+.$$

We write $\omega^0 = H_+$ and, for each $i \in \{1, \dots, N\}$,

$$\omega^i = \underline{\mathrm{Im}}\, P_{i,i-1}|_{\underline{\mathrm{Ker}}\, Q_i|_W} \oplus \lambda^i H_+ \,, \tag{5.14}$$

where $P_{i,l}, Q_i : \underline{E}^N \to \underline{E}^N$, with $i, l \in \{0, \ldots, N\}$, are the holomorphic bundle morphisms defined by

$$\begin{cases} Q_i(\sum \lambda^j a_j) = \sum_{j=-k}^{-i-1} \lambda^j a_j \\ P_{i,l}(\sum \lambda^j a_j) = \sum_{j=-i}^{l} \lambda^j a_j. \end{cases} \tag{5.15}$$

Since W is holomorphic, by Proposition (2) we conclude that each ω^i is a holomorphic subbundle of $\underline{E}^N$. Moreover, note that ω^i coincide with W^i almost everywhere. Since each W^i has values in $\mathrm{Gr}_{\mathbb{H}}^{(n)}$ and satisfies $W^i \subset \lambda^{-1} W^{i+1}$, we conclude by continuity that ω^i also has values in $\mathrm{Gr}_{\mathbb{H}}^{(n)}$ and $\omega^i \subset \lambda^{-1}\omega^{i+1}$. Hence the sequence

$$H_+ = \omega^0, \ldots, \omega^N = \Phi \cdot H_+$$

defines a new factorization of Φ,

$$\Phi = \Phi_1 \ldots \Phi_N \,,$$

with

$$\omega^i = \Phi_1 \ldots \Phi_i \cdot H_+ : M \to \Omega^i_{\mathrm{alg}} \mathrm{Sp}(n) \,.$$

Lemma 18. Each ω^i is a complex extended solution.

Proof. We have already seen that ω^i is holomorphic, that is, $(\omega^i)_{\bar{z}} \subset \omega^i$. On the other hand, off a discrete set of points D, ω^i is given by

$$W^i = (W \cap \lambda^{-i} H_+) + \lambda^i H_+ \,.$$

So we can use this formula to compute the derivatives of ω^i on $M \setminus D$. Since W is a complex extended solution, $W_z \subset \lambda^{-1} W$. Hence we have:

$$\begin{aligned} (W^i)_z &= (W_z \cap \lambda^{-i} H_+) + \lambda^i H_+ \\ &\subset (\lambda^{-1} W \cap \lambda^{-i} H_+) + \lambda^i H_+ \\ &= \lambda^{-1}\{(W \cap \lambda^{-i+1} H_+) + \lambda^{i+1} H_+\} \\ &\subset \lambda^{-1}\{(W \cap \lambda^{-i} H_+) + \lambda^{i+1} H_+\} \\ &= \lambda^{-1} W^i \,; \end{aligned}$$

applying continuity we conclude that $(\omega^i)_z \subset \lambda^{-1}\omega^i$, and we are done. □

Hence we have:

Proposition 7. Any extended solution $\Phi : M \to \Omega^N_{\mathrm{alg}}\mathrm{Sp}(n)$ admits a factorization $\Phi = \Phi_1\Phi_2 \dots \Phi_N$, where each Φ_i is a smooth map into $\Omega^1_{\mathrm{alg}}\mathrm{Sp}(n)$ and each subproduct $\Phi_1\Phi_2 \dots \Phi_i : M \to \Omega^i_{\mathrm{alg}}\mathrm{Sp}(n)$ is an extended solution.

There exits a canonical refinement of such a factorization into "linear factors":

We have already seen in section 5.1.2 that the sequence

$$H_+ = W^0(z), W^{0^+}(z), W^1(z), W^{1^+}(z), \dots, W^N(z) = \gamma H_+$$

induces a factorization of $\Phi(z)$, with $z \in M$, into "linear factors", where

$$W^{i^+} = (\lambda W \cap \lambda^{-i}H_+) + \lambda^{i+1}H_+ \, .$$

For each $i \in \{0, \dots, N-1\}$, we write

$$\omega^{i^+} = \underline{\mathrm{Im}}\, P_{i,i}|_{\underline{\mathrm{Ker}}\, Q_i|_{\lambda W}} \oplus \lambda^{i+1}H_+ \, . \tag{5.16}$$

Since λW is holomorphic, each ω^{i^+} is also holomorphic. Note that ω^{i^+} coincides with W^{i^+} almost everywhere. Define F_{i^+} and F_i by setting $\omega^{i^+} = F_{i^+} \cdot H_+$ and $\omega^i = F_i \cdot H_+$. Since W^{i^+} satisfies (5.3), we conclude by continuity that

$$\lambda\omega^i \subset \omega^{i^+} \subset \omega^i$$
$$\lambda\omega^{i^+} \subset \lambda\omega^{i+1} \subset \omega^{i^+} \, .$$

Then $F_{i^+} = F_i\gamma_{V_i}$ and $F_{i+1} = F_{i^+}\gamma_{\tilde{V}_i}{}^{-1}$ for some smooth vector subbundles $V_i, \tilde{V}_i$ of $\underline{\mathbb{C}}^{2n}$, where, for any vector subbundle V of $\underline{\mathbb{C}}^{2n}$, we denote $\gamma_V = \pi_V + \lambda\pi_{V^\perp}$. It follows that the sequence

$$H_+ = \omega^0, \omega^{0^+}, \omega^1, \omega^{1^+}, \dots, \omega^N = \gamma H_+$$

induces a factorization of Φ,

$$\Phi = \underbrace{\gamma_{V_0}\gamma_{\tilde{V}_0}{}^{-1}}_{\Phi_1}\underbrace{\gamma_{V_1}\gamma_{\tilde{V}_1}{}^{-1}}_{\Phi_2} \dots \underbrace{\gamma_{V_{N-1}}\gamma_{\tilde{V}_{N-1}}{}^{-1}}_{\Phi_N}, \tag{5.17}$$

with

$$\omega^{i^+} = F_{i^+} \cdot H_+ = \Phi_1 \dots \Phi_{i-1}\gamma_{V_i} \cdot H_+ \, .$$

Lemma 19. Each ω^{i^+} is a complex extended solution.

Proof. We have already seen that ω^{i^+} is holomorphic, that is, $(\omega^{i^+})_{\bar z} \subset \omega^i$. On the other hand, off a discrete set of points D, ω^{i^+} is given by

$$W^{i^+} = (\lambda W \cap \lambda^{-i} H_+) + \lambda^{i+1} H_+ .$$

So we can use this formula to compute the derivatives of ω^i on $M \setminus D$. Since W is a complex extended solution, $W_z \subset \lambda^{-1} W$. Hence we have:

$$\begin{aligned} (W^{i^+})_z &= (\lambda W_z \cap \lambda^{-i} H_+) + \lambda^{i+1} H_+ \\ &\subset (W \cap \lambda^{-i} H_+) + \lambda^{i+1} H_+ \\ &= \lambda^{-1}\{(\lambda W \cap \lambda^{-i-1} H_+) + \lambda^{i} H_+\} \\ &\subset \lambda^{-1}\{(\lambda W \cap \lambda^{-i} H_+) + \lambda^{i+1} H_+\} \\ &= \lambda^{-1} W^{i^+} ; \end{aligned}$$

applying continuity we conclude that $(\omega^{i^+})_z \subset \lambda^{-1}\omega^{i^+}$, and we are done. □

To summarize:

Theorem 31. Any extended solution $\Phi : M \to \Omega^N_{\mathrm{alg}}\mathrm{Sp}(n)$ admits a factorization of the form (5.17), where V_i and $\tilde V_i$ are J-isotropic smooth vector subbundles of $\underline{\mathbb{C}}^{2n}$; each Φ_i is a smooth map into $\Omega^1_{\mathrm{alg}}\mathrm{Sp}(n)$; and

$$\begin{aligned} F_i &= \Phi_1\Phi_2 \ldots \Phi_i : M \to \Omega^i_{\mathrm{alg}}\mathrm{Sp}(n) \\ F_{i^+} &= \Phi_1 \ldots \Phi_{i-1}\gamma_{V_i} : M \to \Omega_{\mathrm{alg}}\mathrm{U}(2n) \end{aligned}$$

are extended solutions.

Remark. In contrast with the factorization given by Burstall and Rawnsley [12], in general our factors $\gamma_{V_i}(-1)$ and $\gamma^{-1}_{\tilde V_i}(-1)$ do not have values in an Hermitian symmetric $\mathrm{Sp}(n)$-space.

5.3 Harmonic maps into compact inner symmetric $\mathrm{Sp}(n)$-spaces

Proposition 8. [10] Let G be a compact (connected) Lie group. Then each connected component of $\sqrt{e} = \{g \in G \,:\, g^2 = e\}$ is a compact inner symmetric space. Moreover, the embedding of each component of $\sqrt{e}$ in G is totally geodesic.

Examples. 1. If $G = \mathrm{U}(n)$, then the connected components of $\sqrt{e}$ are the inner symmetric spaces $\mathrm{Gr}_k(\mathbb{C}^n)$ for $k = 0, 1, \ldots, n$ and

$$\iota_k : \mathrm{G}_k(\mathbb{C}^n) \to \sqrt{e} \subset \mathrm{U}(n), \quad \iota_k(V) = \pi_V^{\perp} - \pi_V \tag{5.18}$$

are totally geodesic embeddings.

2. Similarly, if $G = \mathrm{Sp}(n)$, then the connected components of $\sqrt{e}$ are the inner symmetric spaces $\mathrm{Gr}_k(\mathbb{H}^n)$ (the Grassmannian of J-stable vector subspaces of $\mathbb{C}^{2n} \cong \mathbb{H}^n$ with quaternionic dimension k) for $k = 0, 1, \ldots, n$, with totally geodesic embeddings defined in same way as in (5.18).

As in Uhlenbeck [44] and Segal [38], we may characterize the corresponding special extended solution in terms of the involution

$$I : \Omega G \to \Omega G, \quad I(\gamma)(\lambda) = \gamma(-\lambda)\gamma(-1)^{-1}.$$

We write

$$(\Omega G)_I = \{\gamma \in \Omega G : \ I(\gamma) = \gamma\}$$

for the fixed set of I.

Proposition 9. [10] i) Let M be a Riemann surface and $\Phi : M \to (\Omega G)_I$ an extended solution. Then $\phi = \Phi_{-1}$ defines a harmonic map from M into (a connected component of) $\sqrt{e}$.

ii) Let $\phi : M \to \sqrt{e}$ be a harmonic map and $\tilde{\Phi} : M \to \Omega G$ an extended solution such that $\phi = \tilde{\Phi}_{-1}$. Then there exists an extended solution $\Phi : M \to (\Omega G)_I$ such that $\phi = \Phi_{-1}$.

Under the identification $\Omega\mathrm{U}(n) \cong \mathrm{Gr}^{(n)}$, I induces an involution on $\mathrm{Gr}^{(n)}$, that we shall also denote by I, and $(\Omega\mathrm{U}(n))_I$ can be identified with

$$\mathrm{Gr}_I^{(n)} = \{W \in \mathrm{Gr}^{(n)} : \text{ if } s(\lambda) \in W \text{ then } s(-\lambda) \in W\}.$$

Let M be a Riemann surface and $\phi : M \to \mathrm{Gr}_m(\mathbb{C}^n)$ a harmonic map. Corresponding to the harmonic map $\iota_m \circ \phi$, there is a complex extended solution $W : M \to \mathrm{Gr}_I^{(n)}$. We write

$$W = W^{\mathrm{even}} \oplus W^{\mathrm{odd}},$$

where "even" and "odd" mean that I acts as $+1$ and -1, respectively. If $W(z) \ominus \lambda W(z)$ is identified with $\mathbb{C}^n$ by evaluating at $\lambda = 1$, then the element of order 2 in $\mathrm{U}(n)$ corresponding to $W(z)$ is given by the decomposition:

$$W(z) \ominus \lambda W(z) = (W(z) \ominus \lambda W(z))^{\mathrm{even}} \oplus (W(z) \ominus \lambda W(z))^{\mathrm{odd}}.$$

Since $\iota_k(V) = \pi_V^\perp - \pi_V$, we must have $\dim(W(z) \ominus \lambda W(z))^{\text{odd}} = m$ and evaluating at $\lambda = 1$

$$(W(z) \ominus \lambda W(z))^{\text{odd}}$$

we recover $\phi(z)$.

The splitting (5.11) is stable under I, and I acts on A_i as $(-1)^i$. We therefore have

$$\sum_{i \text{ odd}} \dim A_i = m\,.$$

Let M be a compact Riemann surface, $\phi : M \to \mathrm{Gr}_m(\mathbb{H}^n)$ a harmonic map and suppose that $\iota_m \circ \phi$ admits a complex extended solution

$$W : S^2 \to \mathrm{Gr}^{(n)}_{\mathbb{H},I}\,.$$

We write $H_+ = H_+^{(2n)}$. Let k be the least integer such that

$$\lambda^k H_+ \subset W(z) \subset \lambda^{-k} H_+$$

for all $z \in M$. Recall that all factors A_i in splitting (5.11) have constant dimension off a discrete set D. Fix an integer $i > 0$ such that $A_i(z) = 0$ for all $z \in M \setminus D$. Hence we also have $A_{-i}(z) = 0$ for all $z \in M \setminus D$. Set

$$V = \lambda^{i-1}(W \cap \lambda^{-i-1} H_+) + \lambda^{1-i}(W \cap \lambda^{i-1} H_+) + \lambda^2 H_+\,, \tag{5.19}$$

which is a well defined holomorphic map from M to $\mathrm{Gr}_I^{(2n)}$ except perhaps at finite set of points.

Lemma 20. *V is a constant element of $\mathrm{Gr}^{(n)}_{\mathbb{H},I}$.*

Proof. Once again, using relations (5.2) we have:

$$\begin{aligned}
JV^\perp &= \{\lambda^{-i+1}(JW \cap \lambda^{i+1} JH_+) + \lambda^{-1+i}(JW \cap \lambda^{1-i} JH_+) + \lambda^{-2} JH_+\}^\perp \\
&= \{\lambda^{-i+1}(\lambda W^\perp \cap \lambda^{i+2} H_+^\perp) + \lambda^{-1+i}(\lambda W^\perp \cap \lambda^{2-i} H_+^\perp) + \lambda^{-1} H_+^\perp\}^\perp \\
&= \{\lambda^{-i+2} W^\perp \cap \lambda^3 H_+^\perp\}^\perp \cap \{\lambda^i W^\perp \cap \lambda H_+^\perp\}^\perp \cap \{\lambda^{-1} H_+\} \\
&= \{\lambda^{-i+2} W + \lambda^3 H_+\} \cap \{\lambda^i W + \lambda H_+\} \cap \{\lambda^{-1} H_+\} \\
&= \{\lambda^{-i+2} W + \lambda^3 H_+\} \cap \{\lambda^i W \cap \lambda^{-1} H_+ + \lambda H_+\} \\
&= \lambda^3 H_+ + \lambda^{-i+2} W \cap \{\lambda^i W \cap \lambda^{-1} H_+ + \lambda H_+\} \\
&= \lambda^3 H_+ + \lambda^i W \cap \lambda^{-1} H_+ + \lambda^{-i+2} W \cap \lambda H_+ \\
&= \lambda V\,.
\end{aligned}$$

Hence $V \in \mathrm{Gr}_{\mathbb{H},I}$ and

$$\lambda^2 H_+ \subset V \subset \lambda^{-2} H_+ \,. \tag{5.20}$$

We prove now that V is constant, or equivalently, that V is both holomorphic and anti-holomorphic. Since W is a complex extended solution, it satisfies equations (5.6) and (5.7). From (5.7) we see immediately that

$$V_{\bar{z}} \subset V \,,$$

that is, V is holomorphic. From (5.6) results that

$$V_z \subset \lambda^{-1} V \,. \tag{5.21}$$

Define $V_l = p_l(V \cap \lambda^l H_+)$. Then

$$V_{-1} = p_{-1}(V \cap \lambda^{-1} H_+) = p_{-i}(W \cap \lambda^{-i} H_+) = W_{-i} \,,$$

and

$$V_{-2} = p_{-2}(V \cap \lambda^{-2} H_+) = p_{-i-1}(W \cap \lambda^{-i-1} H_+) = W_{-i-1} \,,$$

hence $V_{-1}/V_{-2} = W_{-i}/W_{-i-1} = 0$. In same way, $V_1/V_0 = W_i/W_{i-1} = 0$. Then, since (5.20) holds, we have:

$$(V \ominus \lambda V)^{\mathrm{odd}} \cong V_{-1}/V_{-2} \oplus V_1/V_0 = 0 \,.$$

V can be decomposed into $+1$ and -1-eigenspaces of I:

$$V = V_+ \oplus V_- \,.$$

Note that

$$V_- \ominus \lambda V_+ = (V \ominus \lambda V)^{\mathrm{odd}} = 0 \,;$$

thus, using (5.21), we have

$$V_{+z} \subset \lambda^{-1} V_- = V_+$$

and

$$V_{-z} = \lambda V_{+z} \subset \lambda V_+ = V_- \,,$$

which means that $V_z \subset V$, hence V is also holomorphic. $\square$

Since $V \in \mathrm{Gr}^{(n)}_{\mathbb{H},I}$ and $(V \ominus \lambda V)^{\mathrm{odd}} = 0$, defining the constant loop γ in $(\Omega \mathrm{Sp}(n))_I$ by $V = \gamma \cdot H_+$, we see that $\gamma(-1) = \mathrm{Id}$. Suppose that Φ is the extended solution associated with W. Then, since

$$\lambda^{k-2} V \subset W \subset \lambda^{2-k} V \,,$$

we have

$$\lambda^{k-2} H_+ \subset \gamma^{-1}\Phi \cdot H_+ \subset \lambda^{2-k} H_+ \,,$$

with $\gamma^{-1}\Phi$ being an extended solution associated to the same original map ϕ, because γ is constant and $\gamma(-1) = \mathrm{Id}$. Iterating this process we conclude:

Theorem 32. Let M be a compact Riemann surface and $\phi : M \to \mathrm{Gr}_m(\mathbb{H}^n)$ a harmonic map. Suppose that $\iota_m \circ \phi$ admits a complex extended solution. Then $\iota_m \circ \phi$ admits a complex extended solution $W : M \to \mathrm{Gr}^{(n)}_{\mathbb{H},I}$ such that

$$\lambda^{2m} H_+ \subset W \subset \lambda^{-2m} H_+$$

(in this case we say that W is a *normalized complex extended solution* associated to ϕ).

In particular, any harmonic map from the sphere to $\mathbb{H}P^{n-1}$ has a complex extended solution W such that $\lambda^2 H_+ \subset W \subset \lambda^{-2} H_+$.

5.3.1 A factorization theorem for harmonic maps into a quaternionic Grassmannian

Let M be a Riemann surface and $\phi : M \to \mathrm{Gr}_m(\mathbb{H}^n)$ a harmonic map admitting an extended solution $\Phi : M \to (\Omega^N_{\mathrm{alg}}\mathrm{Sp}(n))_I$. Let $W : M \to \mathrm{Gr}^{(n)}_{\mathbb{H},I}$ correspond to Φ under the identification $(\Omega\mathrm{Sp}(n))_I \cong \mathrm{Gr}^{(n)}_{\mathbb{H},I}$, $W = \Phi \cdot H_+$. Note that the associated holomorphic maps ω^i and ω^{i^+} defined by formulas (5.14) and (5.16), respectively, also have values in $\mathrm{Gr}^{(n)}_{\mathbb{H},I}$. Hence, we have the following version of Theorem 31:

Theorem 33. Any extended solution $\Phi : M \to (\Omega^N_{\mathrm{alg}}\mathrm{Sp}(n))_I$ admits a factorization of the form (5.17), where V_i and $\tilde{V}_i$ are J-isotropic smooth vector subbundles of $\underline{\mathbb{C}}^{2n}$; each Φ_i is a smooth map into $\Omega^1_{\mathrm{alg}}\mathrm{Sp}(n)$; and

$$\begin{aligned} F_i &= \Phi_1\Phi_2 \ldots \Phi_i : M \to (\Omega^i_{\mathrm{alg}}\mathrm{Sp}(n))_I \\ F_{i^+} &= \Phi_1 \ldots \Phi_{i-1}\Phi_{V_i} : M \to (\Omega_{\mathrm{alg}}\mathrm{U}(2n))_I \end{aligned}$$

are extended solutions.

5.4 Harmonic spheres in the quaternionic projective space

Let A_{-2} and A_{-1} be two subspaces of $\mathbb{C}^{2n}$ such that

$$A_{-2} \subset A_{-1} \subset JA_{-1}^{\perp} \subset JA_{-2}^{\perp} \,.$$

Let $T_1 : A_{-2} \to JA_{-1}$ be a complex linear map such that

$$R_{T_1} = \{v = a + b \in \mathbb{C}^{2n} : \ b = T_1(a),\, a \in A_{-2}\}$$

satisfies $R_{T_1} \perp JR_{T_1}$. Define

$$\mathcal{U}_{T_1} = \{s \in H^{(2n)} : \ s(\lambda) = \lambda^{-2}a + T_1(a) \text{ with } a \in A_{-2}\} \quad (5.22)$$
$$\mathcal{U}_{T_2} = \{s \in H^{(2n)} : \ s(\lambda) = \lambda^{-1}b + \lambda T_2(b) \text{ with } b \in A_{-1}\} \quad (5.23)$$

where $T_2 = JT_1^*J : A_{-1} \to JA_{-2}$.

Proposition 10. a) The subspace W of $H^{(2n)}$ defined by

$$W = \mathcal{U}_{T_1} \oplus JA_{-1}^{\perp} \oplus \mathcal{U}_{T_2} \oplus \lambda JA_{-2}^{\perp} \oplus \lambda^2 H_+\,.$$

is an element of $\mathrm{Gr}^{(n)}_{\mathbb{H},I}$.

b) Conversely, any $W \in \mathrm{Gr}^{(n)}_{\mathbb{H},I}$ with $\lambda^2 H_+ \subset W \subset \lambda^{-2}H_+$ arises in this way.

Proof. a) Clearly W lies on the fixed set of the involution I. In order to prove that $\lambda W \subset W$, the only non-trivial detail we have to check is that $\lambda\mathcal{U}_{T_1} \subset \mathcal{U}_{T_2} \oplus \lambda JA_{-2}^{\perp}$: since $R_{T_1} \perp JR_{T_1}$, for any $s_a = a + T_1(a), s_b = b + T_1(b) \in R_{T_1}$ we have

$$\begin{aligned} 0 = \langle s_a, Js_b\rangle &= \langle a + T_1(a), Jb + JT_1(b)\rangle \\ &= \langle a, JT_1(b)\rangle + \langle T_1(a), Jb\rangle \\ &= -\langle a, \underbrace{JT_1J}_{=T_2^*}(Jb)\rangle + \langle T_1(a), Jb\rangle \\ &= -\langle T_2(a), Jb\rangle + \langle T_1(a), Jb\rangle\,, \end{aligned}$$

which means that $\pi_{JA_{-2}} \circ T_1 = T_2 \circ \pi_{A_{-2}}$; whence

$$\begin{aligned} \underbrace{\lambda^{-1}a + \lambda T_1(a)}_{\in \lambda\mathcal{U}_{T_1}} &= \lambda^{-1}a + \lambda\pi_{JA_{-2}}(T_1(a)) + \lambda\pi_{JA_{-2}^{\perp}}(T_1(a)) \\ &= \underbrace{\lambda^{-1}a + \lambda T_2(a)}_{\in\mathcal{U}_{T_2}} + \underbrace{\lambda\pi_{JA_{-2}^{\perp}}(T_1(a))}_{\in\lambda JA_{-2}^{\perp}}\,, \end{aligned}$$

and we are done.

Finally we have to prove that $JW^{\perp} = \lambda W$. Note that

$$\begin{aligned} JW^{\perp} &= \{(A_{-1} \oplus \lambda^2 JA_{-2}) \ominus J\mathcal{U}_{T_1}\} \oplus \lambda^2 JA_{-2}^{\perp} \\ &\quad \oplus \{(\lambda^{-1}A_{-2} \oplus \lambda JA_{-1}) \ominus J\mathcal{U}_{T_2}\} \oplus \lambda JA_{-1}^{\perp} \oplus \lambda^3 H_+\,; \end{aligned}$$

whence $JW^{\perp} = \lambda W$ if and only if

$$\begin{cases} \lambda \mathcal{U}_{T_1} = (\lambda^{-1} A_{-2} \oplus \lambda J A_{-1}) \ominus J\mathcal{U}_{T_2} \\ \lambda \mathcal{U}_{T_2} = (A_{-1} \oplus \lambda^2 J A_{-2}) \ominus J\mathcal{U}_{T_1} \,. \end{cases} \tag{5.24}$$

We observe that these equations are equivalent, then we just have to check one of them. Take $s_a = \lambda^{-1} a + \lambda T_1(a) \in \lambda \mathcal{U}_{T_1}$ and $s_b = \lambda J b + \lambda^{-1} J T_2(b) \in J\mathcal{U}_{T_2}$. Since $T_2^* = J T_1 J$, we have

$$\begin{aligned} \langle s_a, s_b \rangle &= \langle \lambda^{-1} a + \lambda T_1(a), \lambda J b + \lambda^{-1} J T_2(b) \rangle \\ &= \langle a, J T_2(b) \rangle + \langle T_1(a), J b \rangle \\ &= \langle a, J T_2(b) \rangle + \langle a, T_1^*(J b) \rangle \\ &= \langle a, J T_2(b) \rangle - \langle a, J T_2(b) \rangle = 0 \,, \end{aligned}$$

which means that $\lambda \mathcal{U}_{T_1} \subset (\lambda^{-1} A_{-2} \oplus \lambda J A_{-1}) \ominus J\mathcal{U}_{T_2}$. Since $\dim \mathcal{U}_{T_1} = \dim A_{-2}$ and $\dim \mathcal{U}_{T_2} = \dim A_{-1}$, we conclude that the equality holds and so $JW^{\perp} = \lambda W$.

b) Suppose that $W \in \mathrm{Gr}^{(n)}_{\mathbb{H},I}$ satisfies $\lambda^2 H_+ \subset W \subset \lambda^{-2} H_+$. We can write

$$W = \mathcal{U}^+ \oplus \mathcal{U}^- \oplus \lambda^2 H_+ \,,$$

where $\mathcal{U}^+ \subset \lambda^{-2}\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ and $\mathcal{U}^- \subset \lambda^{-1}\mathbb{C}^{2n} \oplus \lambda \mathbb{C}^{2n}$. Define

$$A_i = \{ a \in \mathbb{C}^{2n} : \ a = p_i(s(\lambda)) \text{ for some } s(\lambda) \in W \} \,,$$

where, as before, $p_i(\sum \lambda^j a_j) = a_i$. By definition,

$$W \subset \lambda^{-2} A_{-2} \oplus \lambda^{-1} A_{-1} \oplus A_0 \oplus \lambda A_1 \oplus \lambda^2 H_+ \,;$$

then

$$\lambda^{-2} J A_1^{\perp} \oplus \lambda^{-1} J A_0^{\perp} \oplus J A_{-1}^{\perp} \oplus \lambda J A_{-2}^{\perp} \oplus \lambda^2 H_+ \subset \lambda^{-1} J W^{\perp} = W \,. \tag{5.25}$$

One can easily check that (5.25) gives the maximal split subspace contained in W, hence there are two well defined complex linear maps

$$\begin{aligned} T_1 &: A_{-2} \to J A_{-1} \\ T_2 &: A_{-1} \to J A_{-2} \,, \end{aligned}$$

where, for $a \in A_{-2}$, $T_1(a)$ is the only vector of JA_{-1} such that $\lambda^{-2} a + T_1(a) \in \mathcal{U}^+$ and, for $b \in A_{-1}$, $T_2(b)$ is the only vector of JA_{-2} such that $\lambda^{-1} b + \lambda T_2(b) \in \mathcal{U}^-$. Define $\mathcal{U}_{T_1}, \mathcal{U}_{T_2}$ as in (5.22) and (5.23). Then

$$W = \underbrace{\mathcal{U}_{T_1} \oplus J A_{-1}^{\perp}}_{\mathcal{U}^+} \oplus \underbrace{\mathcal{U}_{T_2} \oplus \lambda J A_{-2}^{\perp}}_{\mathcal{U}^-} \oplus \lambda^2 H_+ \,.$$

Reversing straightforwardly the arguments given to prove part a) of our Proposition, one can easily see that

$$\begin{cases} A_{-2} \subset A_{-1} \subset JA_{-1}^{\perp} \subset JA_{-2}^{\perp} \\ R_{T_1} \perp JR_{T_1} \\ T_2 = JT_1^* J\,. \end{cases}$$

□

Now we shall establish our classification of harmonic spheres in the quaternionic projective space:

Theorem 34. i) Let M be a Riemann surface and $A \subset B$ be two holomorphic J-isotropic vector subbundles of $M \times \mathbb{C}^{2n}$ such that $\dim B \ominus A = 1$. Suppose also that $A_z \subset B$ and $B_z \subset JB^{\perp}$. Let R be a holomorphic vector subbundle of $A \oplus JB$ with $\dim R = \dim A$ such that:

a) $\partial s \perp JR$ for all $s \in C^{\infty}(R)$ (in particular, $R \perp JR$);

b) $\pi_A(R) = A$ almost everywhere.

Then

$$\underline{\phi} = (B \oplus JB) \ominus (R \oplus JR) \tag{5.26}$$

gives a harmonic map $\phi : M \to \mathbb{H}P^{n-1}$.

ii) Conversely, any harmonic map $\phi : S^2 \to \mathbb{H}P^{n-1}$ arises in this way.

Proof. ii) Suppose that $\phi : S^2 \to \mathbb{H}P^{n-1}$ is harmonic and $W : S^2 \to \mathrm{Gr}_{\mathbb{H},I}^{(n)}$ is a normalized complex extended solution corresponding to ϕ:

$$W = \mathcal{U}_{T_1} \oplus JA_{-1}^{\perp} \oplus \mathcal{U}_{T_2} \oplus \lambda JA_{-2}^{\perp} \oplus \lambda^2 H_+\,.$$

Since W is holomorphic,

$$A = \underline{\mathrm{Im}}\, p_{-2}|_W,\ \ B = \underline{\mathrm{Im}}\, p_{-1}|_W \text{ and } C = \underline{\mathrm{Im}}\,(p_{-2} + p_0)|_W\,,$$

where p_i are the holomorphic bundle morphisms defined in (5.9), are holomorphic subbundles of $\underline{\mathbb{C}}^{2n}$ which coincide with A_{-2}, A_{-1} and $R_{T_1} \oplus JA_{-1}^{\perp}$, respectively, almost everywhere. Since $A_{-2} \subset A_{-1} \subset JA_{-1}^{\perp} \subset R_{T_1} \oplus JA_{-1}^{\perp}$, we conclude by continuity that $A \subset B \subset JB^{\perp} \subset C$. In particular, $R = C \ominus JB^{\perp}$ is a well defined (smooth) subbundle of $\underline{\mathbb{C}}^{2n}$ which coincide with R_{T_1} almost everywhere. Hence, $\pi_A(R) = A$ almost everywhere and $\dim R = \dim A$.

Now, since $W_z \subset \lambda^{-1}W$, we obtain

$$\begin{array}{rcl} A_z & \subset & B \\ B_z & \subset & JB^{\perp} . \end{array}$$

In order to describe the derivatives of R we start to observe that

$$\begin{array}{rcl} \tilde{\mathcal{U}}_{T_1} & = & \underline{\mathrm{Im}}\,(\lambda^{-2}p_{-2} + p_0) \ominus JB^{\perp} \\ \tilde{\mathcal{U}}_{T_2} & = & \underline{\mathrm{Im}}\,(\lambda^{-1}p_{-1} + \lambda p_1) \ominus JA^{\perp} , \end{array}$$

are well defined (smooth) vector subbundles of $\underline{E}^2$, which coincide with $\mathcal{U}_{T_1}$ and $\mathcal{U}_{T_2}$, respectively, off a discrete set of points. Since $\tilde{\mathcal{U}}_{T_1} \subset \lambda^{-2}B \oplus JB$, B is holomorphic, and $B_z \subset JB^{\perp}$ is equivalent to $(JB)_{\bar{z}} \subset B^{\perp}$, we have

$$(\tilde{\mathcal{U}}_{T_1})_{\bar{z}} \subset \lambda^{-2}B \oplus JB \oplus (JB^{\perp} \ominus B) .$$

At same time, since $W_{\bar{z}} \subset W$,

$$(\tilde{\mathcal{U}}_{T_1})_{\bar{z}} \subset \tilde{\mathcal{U}}_{T_1} \oplus JB^{\perp} .$$

Then

$$(\tilde{\mathcal{U}}_{T_1})_{\bar{z}} \subset \tilde{\mathcal{U}}_{T_1} \oplus (JB^{\perp} \ominus B) .$$

Evaluating at $\lambda = 1$ one sees that

$$(R)_{\bar{z}} \subset R \oplus (JB^{\perp} \ominus B) ,$$

that is, R is holomorphic in $A \oplus JB$. Finally, since $\tilde{\mathcal{U}}_{T_1} \subset \lambda^{-2}A \oplus JB$, JB is anti-holomorphic and $A_z \subset B$, we have

$$(\tilde{\mathcal{U}}_{T_1})_z \subset \lambda^{-2}B \oplus JB .$$

At same time, since $W_z \subset \lambda^{-1}W$,

$$(\tilde{\mathcal{U}}_{T_1})_z \subset \lambda^{-1}\tilde{\mathcal{U}}_{T_2} \oplus JA^{\perp} .$$

Then

$$(\tilde{\mathcal{U}}_{T_1})_z \subset \lambda^{-1}\tilde{\mathcal{U}}_{T_2} \oplus (JA^{\perp} \ominus JB^{\perp}) .$$

Together with equations (5.24) this gives

$$(\tilde{\mathcal{U}}_{T_1})_z \subset \{(\lambda^{-2}B \oplus JA) \ominus \lambda^{-2}J\tilde{\mathcal{U}}_{T_1}\} \oplus (JA^{\perp} \ominus JB^{\perp}) .$$

Evaluating at $\lambda = 1$ we conclude that $\partial s \perp JR$ for all $s \in C^{\infty}(R)$.

The harmonic map ϕ is recovered by evaluating $(W \ominus \lambda W)^{\text{odd}}$ at $\lambda = 1$. However, it is easier to compute $(W \ominus \lambda W)^{\text{even}}$:

$$\begin{aligned}
(W \ominus \lambda W)^{\text{even}} &= \{\mathcal{U}_{T_1} \oplus JA_{-1}^{\perp} \oplus \lambda^2 \mathbb{C}^{2n}\} \ominus \{\lambda \mathcal{U}_{T_2} \oplus \lambda^2 JA_{-2}^{\perp}\} \\
&= \{\mathcal{U}_{T_1} \oplus JA_{-1}^{\perp} \oplus \lambda^2 JA_{-2}\} \ominus \{\lambda \mathcal{U}_{T_2}\} \\
&= \mathcal{U}_{T_1} \oplus (JA_{-1}^{\perp} \ominus A_{-1}) \oplus J\mathcal{U}_{T_1}
\end{aligned}$$

(we applied (5.24) to obtain the last equality). Then

$$\begin{aligned}
\underline{\phi} &= \{R_{T_1} \oplus JR_{T_1} \oplus (JA_{-1}^{\perp} \ominus A_{-1})\}^{\perp} \\
&= (A_{-1} \oplus JA_{-1}) \ominus (R_{T_1} \oplus JR_{T_1}),
\end{aligned}$$

that is,

$$\underline{\phi} = (B \oplus JB) \ominus (R \oplus JR).$$

Since $\dim R = \dim A$ and ϕ has values in $\mathbb{H}P^{n-1}$, we must have $\dim B \ominus A = 1$.

i) Suppose condition b) holds off some discrete set $D \subset M$. Denote by A_{-2}, A_{-1} and $\tilde{R}$ the restrictions to $M \setminus D$ of A, B and R, respectively. Then $\tilde{R} = R_{T_1}$ for some (smooth) bundle morphism $T_1 : A_{-2} \to JA_{-1}$ and $\tilde{R} \perp J\tilde{R}$. Set

$$T_2 = JT_1^* J : A_{-1} \to JA_{-2}.$$

We can define a (smooth) map $W : M \setminus D \to \mathrm{Gr}^{(n)}_{\mathbb{H},I}$ by

$$W = \mathcal{U}_{T_1} \oplus JA_{-1}^{\perp} \oplus \mathcal{U}_{T_2} \oplus \lambda JA_{-2}^{\perp} \oplus \lambda^2 H_+,$$

where $\mathcal{U}_{T_1}$ and $\mathcal{U}_{T_2}$ are defined as in (5.22) and (5.23), respectively. One can show that W is a complex extended solution associated to the harmonic map

$$\tilde{\phi} : M \setminus D \to \mathbb{H}P^{n-1}, \ \underline{\tilde{\phi}} = (A_{-1} \oplus JA_{-1}) \ominus (R_{T_1} \oplus JR_{T_1})$$

by reversing straightforwardly the arguments used to prove ii). But

$$\underline{\phi} = (B \oplus JB) \ominus (R \oplus JR).$$

is a smooth extension of $\underline{\tilde{\phi}}$ to M; hence $\phi : M \to \mathbb{H}P^{n-1}$ is harmonic. □

Remark. We can give a completely holomorphic description of the data in Theorem 34, for which the only structure on $\mathbb{C}^{2n}$ we have to consider is the symplectic form $\omega = \langle \cdot\,, J\,\cdot \rangle$. For each subspace E of $\mathbb{C}^{2n}$, we denote by E° the polar of E with respect to ω. The conditions on the holomorphic subbundles $A \subset B$ are equivalent to say that: A and B are isotropic; $\dim B/A = 1$; $A_z \subset B$ and $B_z \subset B^\circ$. Now, R can be seen as a holomorphic subbundle of $A \oplus \mathbb{C}^{2n}/B^\circ$ satisfying $\pi_1(R) = A$ almost everywhere, where π_1 is the projection over A with respect to the direct sum $A \oplus \mathbb{C}^{2n}/B^\circ$. This means that R is the graph of some holomorphic bundle morphism $T : A \to \mathbb{C}^{2n}/B^\circ$ almost everywhere. Denote by $T^{ad} : B \to \mathbb{C}^{2n}/A^\circ$ the adjoint map of T with respect to ω and consider the linear operator $\partial : C^\infty(\mathbb{C}^{2n}/B^\circ) \to C^\infty(\mathbb{C}^{2n}/A^\circ)$ defined by $\partial[v] = [\partial v]$. This operator is well defined since $A_z \subset B$ and hence $B_z^\circ \subset A^\circ$. Let $R^{ad} \subset B \oplus \mathbb{C}^{2n}/A^\circ$ be the graph of $-T^{ad}$. One can easily show that condition a) in Theorem 34 is equivalent to say that $\partial(C^\infty(R)) \subset C^\infty(R^{ad})$.

Remark. The holomorphic subbundles $A \subset B$ define an S^1-invariant complex extended solution:

$$W = \lambda^{-2} A \oplus \lambda^{-1} B \oplus JB^\perp \oplus \lambda J A^\perp \oplus \lambda^2 H_+\,,$$

which corresponds to the harmonic map

$$\phi = (B \ominus A) \oplus J(B \ominus A)\,.$$

Aithal's characterization of harmonic two-spheres in $\mathbb{H}P^2$ (cf. [1]) arises now as a Corollary to Theorem 34:

Corollary 2. i) Any non-isotropic irreducible harmonic map $\phi : S^2 \to \mathbb{H}P^2$ is given by

$$\underline{\phi} = (\underline{f}^2 \oplus \underline{f}^3 \oplus R \oplus JR)^\perp\,, \tag{5.27}$$

where each $\underline{f}^i$ is the i^{th}-Gauss bundle of a holomorphic map $f : S^2 \to \mathbb{C}P^5$ which is full and *totally J-isotropic*, that is, $\underline{f}^5 = J\underline{f}$, and R is a holomorphic line subbundle of $\underline{f} \oplus \underline{f}^4 \oplus \underline{f}^5$ satisfying

$$\begin{cases} \partial s \perp JR \text{ for all } s \in C^\infty(R) \\ R \not\subset \underline{f}^4 \oplus \underline{f}^5 \text{ almost everywhere}\,. \end{cases} \tag{5.28}$$

ii) Conversely, given any holomorphic map $f : S^2 \to \mathbb{C}P^5$ which is full and totally J-isotropic, and a holomorphic line subbundle R of $\underline{f} \oplus \underline{f}^4 \oplus \underline{f}^5$ satisfying conditions (5.28), then ϕ defined by (5.27) is a non-isotropic irreducible harmonic map $\phi : S^2 \to \mathbb{H}P^2$.

Proof. i) Let $\phi : S^2 \to \mathbb{H}P^2$ be a non-isotropic irreducible harmonic map associated to the bundles A, B and R:

$$\begin{aligned} \underline{\phi} &= (B \oplus JB) \ominus (R \oplus JR) \\ \underline{\phi}^{\perp} &= (JB^{\perp} \ominus B) \oplus (R \oplus JR)\,. \end{aligned}$$

Since A and B are holomorphic, we know by [21],[38] that $\underline{g} = B \ominus A$ represents a harmonic map into $\mathbb{C}P^5$. Denote by $\underline{g}^i$ the i^{th}-Gauss bundle of $\underline{g}$ and by $\underline{\phi}^i$ the i^{th}-Gauss bundle of $\underline{\phi}$. Since any non-isotropic harmonic map into $\mathbb{H}P^n$ has isotropy order even (cf.[2]), we conclude that ϕ has isotropy order 2. Considering the conditions that A, B and R satisfy (see Theorem 34), one can easily show that

$$\begin{cases} (JR)_z \subset JR \oplus \underline{g}^1 \\ R_z \subset \phi \oplus R \\ (JB^{\perp} \ominus B)_z \subset J\underline{g} \oplus (JB^{\perp} \ominus B) \end{cases} \tag{5.29}$$

and

$$\begin{cases} \underline{\phi}^1 \subset JR \oplus \underline{g}^1 \perp \underline{\phi} \\ \underline{\phi}^2 \subset JR \oplus \underline{g}^1 \oplus \underline{g}^2 \perp \underline{\phi} \\ \underline{\phi}^3 \subset JR \oplus \underline{g}^1 \oplus \underline{g}^2 \oplus \underline{g}^3 \not\perp \underline{\phi}\,. \end{cases} \tag{5.30}$$

Now, if $\dim B = 3$, then $JB^{\perp} \ominus B = 0$ and $\underline{\phi}^i \subset JR$ for all i, which means that ϕ could not be non-isotropic. If $\dim B = 1$, then $R = 0$, $\underline{\phi} = B \oplus JB$ and $\underline{\phi}^1 = \underline{g}^1$, that is, ϕ can not be irreducible in this case. So we must have $\dim B = 2$, $\dim A = 1$ and R is a holomorphic line subbundle of $A \oplus JB$. Note also that $\pi_g(\underline{\phi}) \neq 0$. In fact, if we had $\pi_g(\underline{\phi}) = 0$, then since ϕ is J-stable, $\underline{\phi} = A \oplus JA$ and ϕ could not be irreducible since A is holomorphic. Hence, $\pi_g(\underline{\phi}) \neq 0$, $\pi_{Jg}(\underline{\phi}) \neq 0$ and

$$\begin{cases} JB^{\perp} \ominus B = \underline{g}^1 \oplus \underline{g}^2 \\ \underline{g}^3 = J\underline{g} \end{cases} \tag{5.31}$$

On the other hand, suppose A is constant. This implies $R_z \subset \underline{g}^{\perp}$. Hence

$$\begin{aligned} (\underline{\phi}^{\perp})_z &\subset (JB^{\perp} \ominus B)_z \oplus (JR)_z \oplus R_z \\ &\subset \{J\underline{g} \oplus (JB^{\perp} \ominus B)\} \oplus \{JR \oplus g_1\} \oplus \{(\phi \oplus R) \cap \underline{g}^{\perp}\} \end{aligned}$$

and

$$\pi_{\phi}((\underline{\phi}^{\perp})_z) = \pi_{\phi}(J\underline{g}) + \pi_{\phi}(\phi \cap \underline{g}^{\perp})\,.$$

Since ϕ is irreducible and J-stable, we must have $\underline{\phi} = \pi_\phi((\underline{\phi}^\perp)_z)$ and

$$\begin{aligned} 0 &\neq \langle J\pi_\phi(J\underline{g}), \pi_\phi(\underline{\phi} \cap \underline{g}^\perp)\rangle \\ &= \langle \pi_\phi(\underline{g}), \pi_\phi(\underline{\phi} \cap \underline{g}^\perp)\rangle \\ &= \langle \underline{g}, \underline{\phi} \cap \underline{g}^\perp\rangle \,, \end{aligned}$$

which is impossible; then we conclude that A is not constant, which means that $\underline{f} = A$ is a non-constant holomorphic map into $\mathbb{C}P^5$ with $\underline{f}^1 = \underline{g}$. Considering equations (5.31) we conclude now that f is a full totally J-isotropic holomorphic map, that is, $\underline{f}^5 = J\underline{f}$. Moreover, since $A = \underline{f}$, $B = \underline{f} \oplus \underline{f}^1$, our harmonic map is given by

$$\underline{\phi} = (B \oplus JB) \ominus (R \oplus JR) = (\underline{f}^2 \oplus \underline{f}^3 \oplus R \oplus JR)^\perp \,.$$

ii) Conversely, suppose we have full totally J-isotropic holomorphic map $f : S^2 \to \mathbb{C}P^5$ and a holomorphic line bundle R of $\underline{f} \oplus \underline{f}^4 \oplus \underline{f}^5$ satisfying (5.28). Set $B = \underline{f} \oplus \underline{f}^1$ and $A = \underline{f}$. Then it is clear that A, B and R satisfy the conditions of Theorem 34. Hence ϕ given by formula formula (5.26) is a harmonic map. It remains to prove that ϕ is non-isotropic and irreducible:

Suppose $\pi_{\underline{f}^1}(\underline{\phi}) = 0$. Then since $\underline{\phi}$ is J-stable, we have $\underline{\phi} = \underline{f} \oplus J\underline{f}$ and $R \oplus JR = \underline{f}^1 \oplus \underline{f}^4$; in particular, $R \subset \underline{f}^4$, which contradicts the hypothesis on R. Hence, $\pi_{f^1}(\underline{\phi}) \neq 0$, $\pi_{f^4}(\underline{\phi}) \neq 0$ and we see that

$$\begin{cases} \underline{\phi}^1 \subset JR \oplus \underline{f}^2 \perp \underline{\phi} \\ \underline{\phi}^2 \subset JR \oplus \underline{f}^2 \oplus \underline{f}^3 \perp \underline{\phi} \\ \underline{\phi}^3 \subset JR \oplus \underline{f}^2 \oplus \underline{f}^3 \oplus \underline{f}^4 \not\perp \underline{\phi} ., \end{cases} \tag{5.32}$$

that is, ϕ is non-isotropic of isotropy order 2. On the other hand, in this case we have $\pi_{f^1}(R_z) \neq 0$. Then

$$\begin{aligned} \pi_\phi((\underline{\phi}^\perp)_z) &= \pi_\phi((\underline{f}^2 \oplus \underline{f}^3)_z) + \pi_\phi(R_z) + \underbrace{\pi_\phi((JR)_z)}_{=0} \\ &= \underbrace{\pi_\phi(\underline{f}^4)}_{\neq 0} + \underbrace{\pi_\phi(\pi_{f^1}(R_z))}_{\neq 0} + \pi_\phi(\pi_{\underline{f}^{1\perp}}(R_z)) \,. \end{aligned}$$

Since

$$\begin{aligned} \langle \pi_\phi(\underline{f}^4), \pi_\phi(\pi_{f^1}(R_z))\rangle &= \langle \pi_\phi(J\underline{f}^1), \pi_\phi(\pi_{f^1}(R_z))\rangle \\ &= \langle J\pi_\phi(\underline{f}^1), \pi_\phi(\pi_{f^1}(R_z))\rangle = 0 \,, \end{aligned}$$

we conclude that $\underline{\phi} = \pi_\phi((\underline{\phi}^\perp)_z)$, that is, ϕ is irreducible. □

5.5 Further work

In [2] the authors showed that any non-isotropic harmonic map from S^2 to a quaternionic projective space $\mathbb{H}P^{n-1}$ can be obtained in a unique way from a certain type of holomorphic map $f : S^2 \to \mathbb{C}P^{n-1}$ by certain flag transforms. Moreover, by parameterizing the possible flag transforms by holomorphic maps they obtained a one-to-one correspondence between certain finite sequences of rational functions of one complex variable and all non-isotropic harmonic maps $S^2 \to \mathbb{H}P^{n-1}$; the map corresponding to a given sequence of rational functions can be found by a pure algebraic algorithm. Similar characterizations are available to harmonic two-spheres in $G_2(\mathbb{R}^n)$ [3], $G_k(\mathbb{C}^n)$ [46] and $\mathrm{U}(n)$ [47].

1. When $n-1>2$, it is not clear to the author of this thesis how the classification Theorem 34 is related with the work developed in [2].

2. The Grassmannian model for the orthogonal group $\mathrm{O}(n)$ is given by

   Proposition 11. [35] A subspace $W \in \mathrm{Gr}^n$ corresponds to a loop in $\mathrm{O}(n)$ if and only if it belongs to

   $$\mathrm{Gr}^n_{\mathbb{R}} = \{W \in \mathrm{Gr}^n : \ \overline{W}^{\perp} = \lambda W\}\,.$$

   Therefore one can expect that similar results to the ones we have obtained in the $\mathbb{H}P^{n-1}$ case are available in the $G_2(\mathbb{R}^n)$ case. What can be said about harmonic two-spheres in other symmetric spaces and Lie groups by applying the infinite dimensional Grassmannian methodology?

Chapter 6

Dressing actions, Bianchi-Bäcklund and Darboux Transforms

The Bianchi-Bäcklund transforms, which were introduced by Bianchi [4] for surfaces of positive constant Gauss curvature (CGC), are an extension of the Bäcklund transforms (see [20], §120) for surfaces of negative Gauss curvature. In contrast to the negative case, a Bianchi-Bäcklund transform of a real surface is in general complex. In order to obtain a new real surface with positive Gauss curvature, one has to apply two successive Bianchi-Bäcklund transforms, where the second transform has to be matched to the first in a particular way.

In [31], A. Mahler showed in a constructive way that the classical Bianchi-Bäcklund procedure for obtaining a new real surface $\tilde{f}$ out of an old one f amounts to dressing the extended framing associated to the Gauss map of f, which is an harmonic map, by a certain dressing matrix. In this chapter we shall give an alternative approach to Mahler's work. Following the philosophy developed by Terng and Uhlenbeck [42], we start with certain basic elements of $\mathcal{G}_1$, the *simple factors*, for which the dressing action can be computed explicitly. We show that each single Bianchi-Bäcklund transform corresponds to the dressing action of a certain simple factor. A nice geometrical parameterization of these simple factors is available and we shall see how to relate it with the classical parameterization of Bianchi-Bäcklund transforms. As a consequence we prove in an easier way the result announced by Mahler.

Every CMC surface is isothermic, that is, it admits local conformal curvature line coordinates. Darboux [17] discovered a transformation of isothermic surfaces: the surface and its Darboux transform are characterized by the conditions that they have the same conformal structures and curvature lines and are the enveloping surfaces of a 2-sphere congruence. In [27], Hertrich-Jeromin and Pedit gave an alternative approach to the Darboux transforms: all Darboux transforms of a given isothermic surface are described by a Ricatti type equation. For suitable initial conditions, this equation will produce Darboux transforms of constant mean curvature out of an old constant mean curvature surface. We conclude this chapter by relating Darboux and Bianchi-Bäcklund transformations for CMC surfaces.

6.1 CMC and CGC surfaces

Let Σ be a Riemann surface with conformal coordinate $z = x + iy$ and $f : \Sigma \to \mathbb{R}^3$ an immersion. Let $(\cdot, \cdot)$ be the standard inner product of $\mathbb{R}^3$. The map $\varphi : \Sigma \to S^2$ defined by

$$\varphi = \frac{\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}}{\|\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}\|} \tag{6.1}$$

is called the *Gauss map* of f. The *first* and *second fundamental forms* of f are the quadratic forms given by $\mathrm{I}_f = (\mathrm{d}f, \mathrm{d}f)$ and $\mathrm{II}_f = -(\mathrm{d}f, \mathrm{d}\varphi)$, respectively. If II_f is diagonal with respect to z we say that z is a *curvature line* coordinate. The operator $S_f = -\mathrm{d}f^{-1} \circ \mathrm{d}\varphi : T\Sigma \to T\Sigma$ is called the *shape operator.* This is a symmetric operator, therefore it has two eigenvalues, k_1 and k_2, called the *principal curvatures.* A point p on Σ is *umbilic* if the two principal curvatures at p are the same. The *mean curvature* $H = \frac{k_1+k_2}{2}$ and the *Gauss curvature* $K = k_1 k_2$ can be computed from I_f and II_f in the following way:

$$H = \frac{1}{2}\mathrm{tr}\,(\mathrm{II}_f \cdot \mathrm{I}_f^{-1})\,,\ K = \det \mathrm{II}_f (\det \mathrm{I}_f)^{-1}\,.$$

The *Cayley-Hamilton formula* relates all these objects:

$$(\mathrm{d}\varphi, \mathrm{d}\varphi) - 2H\,\mathrm{II}_f + K\,\mathrm{I}_f = 0 \tag{6.2}$$

An immersion for which H is constant is called a *constant mean curvature* (CMC) immersion. Assume now that $f : \Sigma \to \mathbb{R}^3$ is a conformal CMC immersion. In this case, it is well known that the *Hopf differential*

$$Q_f = -(f_z, \varphi_z)\mathrm{d}z^2$$

is holomorphic and that the umbilic points are precisely the points of Σ where Q_f vanishes; if we apply the Cayley-Hamilton formula we see that $Q_f = 0$ where φ is conformal.

Immersions with K constant are called *constant Gauss curvature* (CGC) immersions.

6.2 CMC and CGC surfaces vs. harmonic maps

The main goal of this section is to give a detailed review of some basic facts relating CMC surfaces, CGC surfaces and harmonic maps. All the following results are well known by several authors, but are not very easy to find in the literature (see however [25]).

Let Σ be a Riemann surface with conformal coordinate $z = x + iy$.

Proposition 12. A smooth map $\varphi : \Sigma \to S^2$ is harmonic if and only if

$$\mathrm{d}(\varphi \times *\mathrm{d}\varphi) = 0 \,. \tag{6.3}$$

Proof. We have:

$$\begin{aligned} \mathrm{d}(\varphi \times *\mathrm{d}\varphi)(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) &= -i(\varphi \times \varphi_{\bar{z}})_z - i(\varphi \times \varphi_z)_{\bar{z}} \\ &= -2i(\varphi \times \varphi_{z\bar{z}}) \,. \end{aligned} \tag{6.4}$$

On the other hand, by straightforward computation of $\tau(\varphi)$, the tension field of φ, it is easy to check that φ is harmonic if and only if $\varphi_{z\bar{z}} \perp T_\varphi S^2$. Therefore, from (6.4) we conclude that φ is harmonic if and only if (6.3) holds. □

Hence, if $\varphi : \Sigma \to S^2$ is harmonic and Σ is simply-connected, there exists $F : \Sigma \to \mathbb{R}^3$ such that $\mathrm{d}F = \varphi \times *\mathrm{d}\varphi$.

Proposition 13. a) F is an immersion if and only if φ is an immersion; in this case, F is a CGC $K = 1$ surface and the conformal structure on Σ is given by the second fundamental form II_F.

b) F has an umbilic point at $p \in \Sigma$ if and only if φ is conformal at p.

Proof. a) It is obvious that F is an immersion if and only if φ is an immersion, since $\mathrm{d}F = \varphi \times *\mathrm{d}\varphi$. In this case, we can consider φ as the Gauss map of F, II_F is clearly definite and

$$\mathrm{II}_F(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = -(F_z, \varphi_z) = -i(\varphi \times \varphi_z, \varphi_z) = 0 \,;$$

hence, the second fundamental form II_F gives the conformal structure on Σ.

We compute now the Gauss curvature of F:

$$\begin{aligned}\det \mathrm{II}_F &= (F_{xx},\varphi)(F_{yy},\varphi)\\ &= -((\varphi\times\varphi_y)_x,\varphi)((\varphi\times\varphi_x)_y,\varphi)\\ &= (\varphi_x\times\varphi_y,\varphi)^2\,,\end{aligned}$$

and

$$\begin{aligned}\det \mathrm{I}_F &= |F_x|^2|F_y|^2-(F_x,F_y)^2=|F_x\times F_y|^2\\ &= |\varphi_x\times\varphi_y|^2=(\varphi_x\times\varphi_y,\varphi)^2\,;\end{aligned}$$

Then $K=\det \mathrm{II}_F(\det \mathrm{I}_F)^{-1}=1$.

b) Finally, suppose that $\mathrm{d}F_p=\alpha\mathrm{d}\varphi_p$ for some $p\in\Sigma$ and $\alpha\in\mathbb{R}$. Then, at p we have

$$(\varphi_z,\varphi_z)=-(F_z,F_z)=-\alpha^2(\varphi_z,\varphi_z);$$

and from this we conclude that p is an umbilic point if and only if φ is conformal at p. □

Conversely:

Proposition 14. Let $F:\Sigma\to\mathbb{R}^3$ be a CGC $K=1$ immersion. Then the corresponding Gauss map $\varphi:\Sigma\to S^2$ is harmonic with respect to the conformal structure on Σ provided by II_F.

Proof. Since $K>0$, the second fundamental form II_F is definite. Take a local conformal coordinate system $z=x+iy$ with respect to II_F. Since

$$0=\mathrm{II}_F(\frac{\partial}{\partial z},\frac{\partial}{\partial z})=-(\varphi_z,F_z)\,,$$

we see that $\mathrm{d}F=\alpha(\varphi\times *\mathrm{d}\varphi)$ for some (smooth) function $\alpha:\Sigma\to\mathbb{R}$.

At same time, $K=\det \mathrm{II}_F(\det \mathrm{I}_F)^{-1}=1$. But

$$\begin{aligned}\det \mathrm{II}_F &= (F_{xx},\varphi)(F_{yy},\varphi)\\ &= -((\varphi\times(\alpha\varphi)_y)_x,\varphi)((\varphi\times(\alpha\varphi_x))_y,\varphi)\\ &= \alpha^2(\varphi_x\times\varphi_y,\varphi)^2\,,\end{aligned}$$

and

$$\begin{aligned}\det \mathrm{I}_F &= |F_x|^2|F_y|^2-(F_x,F_y)^2=|F_x\times F_y|^2\\ &= \alpha^4|\varphi_x\times\varphi_y|^2=\alpha^4(\varphi_x\times\varphi_y,\varphi)^2\,;\end{aligned}$$

then, we must have $\alpha^2=1$, that is, $\mathrm{d}F=\pm\varphi\times *\mathrm{d}\varphi$, which means that $\varphi\times *\mathrm{d}\varphi$ is closed, hence ϕ is harmonic with respect to II_F. □

We shall see now that any harmonic map $\varphi : \Sigma \to S^2$ which is everywhere non-conformal, with Σ simply-connected, is the Gauss map of two conformal CMC immersions, both having mean curvature $H = \frac{1}{2}$.

Lemma 21. Let $\varphi : \Sigma \to S^2$ be a harmonic map and $F : \Sigma \to \mathbb{R}^3$ such that $\mathrm{d}F = \varphi \times *\mathrm{d}\varphi$. Suppose that φ is everywhere non-conformal. Then, $f^{\pm} = F \pm \varphi : \Sigma \to \mathbb{R}^3$ are conformal immersions.

Proof. Suppose that $f^{\pm}$ is not an immersion at $p \in \Sigma$. Let X, Y be an orthonormal positively oriented basis of $T_p\Sigma$ such that $\mathrm{d}f^{\pm}_p(X) = 0$. Then $\mathrm{d}F_p(X) = \mp \mathrm{d}\varphi_p(X)$, and so $\mathrm{d}\varphi_p(X) = \mp\varphi \times \mathrm{d}\varphi_p(Y)$, which means that $(\mathrm{d}\varphi_p(Z), \mathrm{d}\varphi_p(Z)) = 0$, with $Z = X - iY$; but this is impossible because φ is everywhere non-conformal.

The fact that f^+ and f^- are conformal can be checked by a mere computation.

□

Suppose that $f : \Sigma \to \mathbb{R}^3$ is conformal and $\varphi : \Sigma \to S^2$ is the corresponding Gauss map. The first fundamental form of f is given by $\mathrm{I}_f = e^{\omega}(\mathrm{d}x^2 + \mathrm{d}y^2)$, for some smooth function $\omega : \Sigma \to \mathbb{R}$, with respect to local conformal coordinates. Then,

$$H = \frac{1}{2}\mathrm{tr}(\mathrm{II}_f \cdot \mathrm{I}_f^{-1}) = \frac{1}{2}e^{-w}(f_{z\bar{z}}, \varphi)\,,$$

or

$$(f_{z\bar{z}}, \varphi) = 2He^{w}\,. \tag{6.5}$$

Moreover, we have the following:

Lemma 22. If $f : \Sigma \to \mathbb{R}^3$ is conformal, then

$$f_{z\bar{z}} \perp f_z, f_{\bar{z}}\,.$$

This can be easily verified by differentiating $(f_z, f_z) = 0$. Lemma 22 together with equation (6.5) gives:

Corollary 3. If $f : \Sigma \to \mathbb{R}^3$ is conformal,

$$f_{z\bar{z}} = (f_{z\bar{z}}, \varphi)\varphi = 2He^{2w}\varphi = 2H\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}\,.$$

Finally we have:

Proposition 15. Let $\varphi : \Sigma \to S^2$ be a harmonic map and $F : \Sigma \to \mathbb{R}^3$ such that $\mathrm{d}F = \varphi \times *\mathrm{d}\varphi$. Suppose that φ is everywhere non-conformal . Then $f^{\pm} = F \pm \varphi : \Sigma \to \mathbb{R}^3$ are both CMC $H = \frac{1}{2}$ immersions (without umbilic points). Moreover, the Gauss map of f^- (resp. f^+) is φ (resp. $-\varphi$).

Proof. We want to compute $f^{\pm}_{z\bar{z}} = f^{\pm}_{xx} + f^{\pm}_{yy}$. Since $\mathrm{d}F = \varphi \times *\mathrm{d}\varphi$,

$$F_{xx} = \varphi \times \varphi_{xy} + \varphi_x \times \varphi_y$$

and

$$F_{yy} = -\varphi \times \varphi_{xy} + \varphi_x \times \varphi_y;$$

then

$$F_{z\bar{z}} = 2\varphi_x \times \varphi_y = 2F_x \times F_y = \varphi_x \times \varphi_y + F_x \times F_y. \tag{6.6}$$

On the other hand,

$$\begin{aligned} \varphi_{z\bar{z}} &= \varphi_{xx} + \varphi_{yy} = (\varphi \times F_y)_x - (\varphi \times F_x)_y \\ &= \varphi_x \times F_y + \varphi \times F_{xy} - \varphi_y \times F_x - \varphi \times F_{xy} \\ &= \varphi_x \times F_y + F_x \times \varphi_y\,. \end{aligned} \tag{6.7}$$

Equations (6.6) and (6.7) imply

$$f^{\pm}_{z\bar{z}} = \frac{\partial f^{\pm}}{\partial x} \times \frac{\partial f^{\pm}}{\partial y}\,.$$

Then, from Corollary 3 we conclude that f^+ and f^- are both conformal CMC immersions with mean curvature $H = \frac{1}{2}$.

By straightforward computation one can easily check that

$$(\frac{\partial f^{\pm}}{\partial x} \times \frac{\partial f^{\pm}}{\partial y}, \varphi) = 2|\varphi_x||\varphi_y| \sin\theta \mp (|\varphi_x|^2 + |\varphi_y|^2)\,,$$

where θ is the angle formed by φ_x and φ_y. Hence,

$$(\frac{\partial f^{+}}{\partial x} \times \frac{\partial f^{+}}{\partial y}, \varphi) < 0$$

and

$$(\frac{\partial f^{-}}{\partial x} \times \frac{\partial f^{-}}{\partial y}, \varphi) > 0.$$

Moreover, since φ is normal to F, φ is also normal to the parallel surfaces $f^{\pm}$. Then we can conclude that the Gauss map of f^- (resp. f^+), which is given by formula (6.1), is φ (resp. $-\varphi$). $\square$

Conversely:

Proposition 16. Let $f : \Sigma \to \mathbb{R}^3$ be a conformal CMC $H = \frac{1}{2}$ immersion without umbilic points and $\varphi : \Sigma \to S^2$ the corresponding Gauss map. Set $F = f + \varphi : \Sigma \to \mathbb{R}^3$. Then $\mathrm{d}F = \varphi \times *\mathrm{d}\varphi$, which means that φ is harmonic (and everywhere non-conformal).

Proof. Since f is conformal, the Cayley-Hamilton formula (6.2) applied to f gives

$$(F_z, \varphi_z) = (\varphi_z, \varphi_z) + (\varphi_z, f_z) = 0\,, \tag{6.8}$$

hence we see that $\mathrm{d}F = \alpha(\varphi \times *\mathrm{d}\varphi)$, with $\alpha = \pm 1$. If $\alpha = -1$, then f would be a $H = -\frac{1}{2}$ CMC surface (see proof of Proposition 15). Hence, $\alpha = 1$ and we are done. □

When a CMC $H = \frac{1}{2}$ immersion has no umbilic points, it is well known that we can always choose a conformal coordinate $z = x + iy$ with respect to which the second fundamental form of f is diagonal, that is, z is a *conformal curvature line* coordinate on Σ. More precisely, we have

$$\mathrm{I}_f = e^{2\omega}(\mathrm{d}x^2 + \mathrm{d}y^2),\ \mathrm{II}_f = e^{\omega}(\sinh\omega \mathrm{d}x^2 + \cosh\omega \mathrm{d}y^2)\,, \tag{6.9}$$

where $\omega : \Sigma \to \mathbb{R}$ is a solution to the Gauss (sinh-Gordon) equation

$$\triangle\omega + \sinh\omega\cosh\omega = 0\,. \tag{6.10}$$

So, away from the points where ω vanishes, $F = f + \varphi$ is a CGC $K = 1$ immersion and we have

$$\mathrm{I}_F = \cosh^2\omega \mathrm{d}x^2 + \sinh^2\omega \mathrm{d}y^2,\ \mathrm{II}_F = -\sinh\omega\cosh\omega(\mathrm{d}x^2 + \mathrm{d}y^2)\,. \tag{6.11}$$

6.3 Bianchi-Bäcklund transforms

The Bianchi-Bäcklund transforms, which were introduced by Bianchi [4] for positive CGC surfaces, are an extension of the Bäcklund transformation (see [20], §120) for surfaces of negative Gauss curvature. We shall now review briefly this theory, following [39].

Let $F : \mathbb{C} \to \mathbb{R}^3$ be a CGC $K = 1$ surface (possibly degenerate) and $z = x + iy$ a curvature line coordinate on $\mathbb{C}$, with the fundamental forms I_F and II_F given by (6.11), where ω is a solution to the sinh-Gordon equation (6.10). In particular, F has no umbilic points and the coordinate z is conformal for the parallel CMC surface but not for F. Let φ be the corresponding Gauss map. Define the orthonormal frame (e_1, e_2, e_3) by:

$$e_1 = \frac{1}{\cosh\omega}F_x\,,\ \ e_2 = \frac{1}{\sinh\omega}F_y\,,\ \ e_3 = e_1 \times e_2\,. \tag{6.12}$$

Let $\tilde{F}: \mathbb{C} \to (\mathbb{R}^3)^{\mathbb{C}} \cong \mathbb{C}^3$ be a complex surface (possibly degenerate) with Gauss map $\tilde{\varphi}$.

Definition 8. [4],[39] We say that $\tilde{F}$ is a *Bianchi-Bäcklund transformation* of the CGC $K = 1$ surface F if it satisfies the following properties:

1. z is a curvature line coordinate with respect to F and $\tilde{F}$;
2. $(\tilde{F} - F, \varphi) = (\tilde{F} - F, \tilde{\varphi}) = 0$ and $\tilde{F} - F$ has constant length;
3. The normals have a constant angle with each other.

So, suppose that $\tilde{F}$ is a Bianchi-Bäcklund transformation of F. Let $\phi: \mathbb{C} \to \mathbb{C}$ be the angle formed by the tangent line $\tilde{F} - F$ and e_1. Then,

$$\tilde{F} = F + \lambda(\cos\phi e_1 + \sin\phi e_2)$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$. We also have

$$(\varphi, \tilde{\varphi}) = \cos\sigma \quad \varphi \times \tilde{\varphi} = \sin\sigma(\cos\phi e_1 + \sin\phi e_2)\,, \tag{6.13}$$

for some constant angle σ.

Theorem 35. [4],[39] $\tilde{F}$ is a new CGC $K = 1$ surface and

$$\tilde{F} = F + \frac{1}{\sinh\beta}(\cosh\theta e_1 + i\sinh\theta e_2)\,, \tag{6.14}$$

where $\beta \in \mathbb{C}\setminus\{0\}$ is a constant, $\lambda = \frac{1}{\sinh\beta}$, $\cot\sigma = -i\cosh\beta$, and $\theta: \mathbb{C} \to \mathbb{C}$, with $\theta = -i\phi$, is a solution of

$$\begin{cases} \theta_x + i\omega_y & = \sinh\beta\sinh\theta\cosh\omega + \cosh\beta\cosh\theta\sinh\omega \\ i\theta_y + \omega_x & = -\sinh\beta\cosh\theta\sinh\omega - \cosh\beta\sinh\theta\cosh\omega \end{cases}\,. \tag{6.15}$$

The first and second fundamental forms of $\tilde{F}$ are given by

$$\mathrm{I}_{\tilde{F}} = \cosh^2\theta \mathrm{d}x^2 + \sinh^2\theta \mathrm{d}y^2,\ \mathrm{II}_{\tilde{F}} = -\sinh\theta\cosh\theta(\mathrm{d}x^2 + \mathrm{d}y^2)\,.$$

Hence, the classical Bianchi-Bäcklund transformations are determined by two complex numbers: the spectral parameter $\beta \in \mathbb{C} \setminus \{0\}$ and an initial angle $\theta_0 \in \mathbb{C}$.

From the analytical point of view, the Bianchi-Bäcklund transformations may be interpreted as a procedure of obtaining new solutions of the sinh-Gordon equation from an old one. In fact, we have:

Theorem 36. [4],[39] If $\omega: \mathbb{R}^2 \to \mathbb{C}$ is a solution of the sinh-Gordon equation, then any solution θ of the Bianchi-Bäcklund PDEs (6.15) will also solve the sinh-Gordon equation.

We denote by $\mathcal{B}_\beta(F)$ (resp. $\mathcal{B}_\beta(\omega)$) the family of Bianchi-Bäcklund transformations of F (resp. ω) with spectral parameter β and by $\mathcal{B}_{\beta,\theta_0}(F)$ (resp. $\mathcal{B}_{\beta,\theta_0}(\omega)$) the Bianchi-Bäcklund transformation of F (resp. ω) determined by (β, θ_0).

Lemma 23. Suppose that $\beta_1, \beta_2 \in \mathbb{C} \setminus \{0\}$ satisfy

$$\sinh\beta_1 = -\sinh\beta_2 \quad \cosh\beta_1 = -\cosh\beta_2\,.$$

Then $\mathcal{B}_{\beta_1}(F) = \mathcal{B}_{\beta_2}(F)$.

Proof. Let θ_1 be a solution of (6.15) for β_1. Then $\theta_2 = i\pi + \theta_1$ is a solution of (6.15) for β_2. Taking account formula (6.14), we conclude that the pairs (β_1, θ_1) and (β_2, θ_2) produce the same surface, and we are done. □

In contrast to the negative CGC case, the solution θ of the Bianchi-Bäcklund PDEs is in general complex and so $\tilde{F}$ will be complex. To obtain a new real solution of sinh-Gordon equation we must perform two iterations of the Bianchi-Bäcklund transformations.

Theorem 37. [4],[39] Let F be a CGC $K = 1$ surface with fundamental forms (6.11). Suppose that $\theta \in \mathcal{B}_\beta(\omega)$ and $\theta^* \in \mathcal{B}_{\beta^*}(\omega)$. Then $F^* = F + \Lambda(Ae_1 + Be_2 + Ce_3)$ is also a CGC $K = 1$ surface, where

$$\begin{aligned}
\Lambda &= \frac{\sinh(\beta-\beta^*)}{\sinh\beta\sinh\beta^*(\cosh(\beta-\beta^*)-\cosh(\theta-\theta^*))}\\
A &= \cosh\beta\cosh\theta^* - \cosh\beta^*\cosh\theta\\
B &= i\cosh\beta\sinh\theta^* - i\cosh\beta^*\sinh\theta\\
C &= \sinh(\theta-\theta^*).
\end{aligned}$$

Moreover, $F^* \in \mathcal{B}_{\beta^*}(\mathcal{B}_\beta(F))$.

The expression for F^* above specifies a complex surface. However, Bianchi observed that for a special choice of parameters, the two-step procedure will yield real solutions:

Theorem 38. [4],[39] Start with the real CGC $K = 1$ surface F. If the reality condition $\beta^* = i\pi - \overline{\beta}$ and hence $\theta^* = -\overline{\theta}$, holds, then $\mathcal{B}_{\beta^*,\theta_0^*}(\mathcal{B}_{\beta,\theta_0}(F))$ is real.

Finally we recall the Bianchi–Bäcklund Permutativity theorem:

Theorem 39. [4],[39] Let F be a CGC $K = 1$ surface and $\beta, \beta^* \in \mathbb{C} \setminus \{0\}$. Then $\mathcal{B}_{\beta^*}(\mathcal{B}_\beta(F)) = \mathcal{B}_\beta(\mathcal{B}_{\beta^*}(F))$.

6.4 Bobenko-Sym formula

As $\mathrm{SO}(3,\mathbb{R})$-modules, $(\mathbb{R}^3,\times)\cong\mathfrak{so}(3,\mathbb{R})$ via

$$u\in\mathbb{R}^3\mapsto\xi_u\in\mathfrak{so}(3,\mathbb{R}),\ \ \xi_u(v)=u\times v\,.$$

The inner product on $\mathfrak{so}(3,\mathbb{R})$ inherited from $\mathbb{R}^3$ is

$$(\xi,\eta)=-\frac{1}{2}\mathrm{tr}\,\xi\eta\,.$$

Let e_1,e_2,e_3 be the canonical orthonormal basis of $\mathbb{R}^3$ and $K_0\subset\mathrm{SO}(3,\mathbb{R})$ the stabilizer of $e_1\in S^2$. Consider the automorphism τ of $\mathrm{SO}(3,\mathbb{R})$ given by conjugation by

$$Q=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}.$$

Then K_0 is the identity component of the fixed set of τ, K. The corresponding symmetric decomposition $\mathfrak{so}(3,\mathbb{C})=\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{m}^{\mathbb{C}}$ is given by:

$$\mathfrak{k}^{\mathbb{C}}=\{x\xi_{e_1}\,:\,x\in\mathbb{C}\}\,,\ \mathfrak{m}^{\mathbb{C}}=\{y\xi_{e_2}+z\xi_{e_3}\,:\,y,z\in\mathbb{C}\}\,.$$

Inspired by the classical theory of Bianchi-Bäcklund transforms, where we have to deal with complex surfaces, we shall now generalize the notion of extended framing:

Definition 9. A smooth map $E:\mathbb{C}\to\Lambda_\tau\mathrm{SO}(3,\mathbb{C})$ is called a *complex extended framing* if E satisfies

$$E^{-1}\mathrm{d}E=(\lambda^{-1}A+B)\mathrm{d}z+(C+\lambda D)\mathrm{d}\bar{z}\,,$$

where $B,C:\mathbb{C}\to\mathfrak{k}^{\mathbb{C}}$ and $A,D:\mathbb{C}\to\mathfrak{m}^{\mathbb{C}}$.

Of course, an extended framing is a complex extended framing satisfying the reality condition $E(\lambda)=\overline{E(\frac{1}{\bar\lambda})}$.

Suppose now we have a complex extended framing $E:\mathbb{C}\to\Lambda_\tau\mathrm{SO}(3,\mathbb{C})$. Define $\varphi:\mathbb{C}\to\mathfrak{so}(3,\mathbb{C})$ by $\varphi=E_1\cdot\varphi_0$, where $\varphi_0=\xi_{e_1}$.

Proposition 17. (*Bobenko-Sym formula*) The smooth map

$$F=-i\Big(\frac{\partial E}{\partial\lambda}\Big)_{\lambda=1}E_1^{-1}:\mathbb{C}\to\mathfrak{so}(3,\mathbb{C})\tag{6.16}$$

satisfies

$$\mathrm{d}F = [\varphi, *\mathrm{d}\varphi] \ .$$

In particular, F is a (real or complex) CGC $K = 1$ immersion if φ is an immersion.

Proof. Since E is a complex extended framing, we have

$$E^{-1}\frac{\partial E}{\partial z} = \lambda^{-1}A + B \,,$$

where $A : \mathbb{C} \to \mathfrak{m}^{\mathbb{C}}$ and $B : \mathbb{C} \to \mathfrak{k}^{\mathbb{C}}$. Then,

$$\begin{aligned}
\frac{\partial F}{\partial z} &= -i\Big(\frac{\partial}{\partial \lambda}\frac{\partial E}{\partial z}\Big)_{\lambda=1} E_1^{-1} + i\Big(\frac{\partial E}{\partial \lambda}\Big)_{\lambda=1} E_1^{-1}\frac{\partial E_1}{\partial z}E_1^{-1} \\
&= -i\Big(\frac{\partial}{\partial \lambda}EE^{-1}\frac{\partial E}{\partial z}\Big)_{\lambda=1} E_1^{-1} + i\Big(\frac{\partial E}{\partial \lambda}\Big)_{\lambda=1} E_1^{-1}\frac{\partial E_1}{\partial z}E_1^{-1} \\
&= -iE_1\Big(\frac{\partial}{\partial \lambda}E^{-1}\frac{\partial E}{\partial z}\Big)_{\lambda=1} E_1^{-1} \\
&= iE_1AE_1^{-1} \,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\big[\varphi, *\mathrm{d}\varphi(\frac{\partial}{\partial z})\big] &= i\big[\varphi, \frac{\partial \varphi}{\partial z}\big] = i\big[E_1\varphi_0E_1^{-1}, \frac{\partial E_1}{\partial z}\varphi_0E_1^{-1} - E_1\varphi_0E_1^{-1}\frac{\partial E_1}{\partial z}E_1^{-1}\big] \\
&= iE_1\big[\varphi_0, \big[E_1^{-1}\frac{\partial E_1}{\partial z}, \varphi_0\big]\big]E_1^{-1} \\
&= iE_1\big[\varphi_0, \big[A + B, \varphi_0\big]\big]E_1^{-1} \\
&= iE_1AE_1^{-1} \,;
\end{aligned}$$

whence

$$\frac{\partial F}{\partial z} = \big[\varphi, *\mathrm{d}\varphi(\frac{\partial}{\partial z})\big] \,.$$

Similarly, one can prove that

$$\frac{\partial F}{\partial \bar{z}} = \big[\varphi, *\mathrm{d}\varphi(\frac{\partial}{\partial \bar{z}})\big] \,,$$

and we are done. □

6.5 Dressing action

Let G be a compact semisimple Lie group and $\tau : G \to G$ an involution. Let $\mathcal{G}$ denote the group of holomorphic maps $g : \mathrm{dom}(g) \subset \mathbb{P}^1 \to G^{\mathbb{C}}$ of open subsets of the Riemann sphere which are twisted in the sense that $\tau g(\lambda) = g(-\lambda)$, for all $\lambda \in \mathrm{dom}(g)$. We define the automorphism $\mathcal{R} : \mathcal{G} \to \mathcal{G}$ by

$$\mathcal{R}(g)(\lambda) = \overline{g\Big(\frac{1}{\overline{\lambda}}\Big)}$$

and the subgroups $\mathcal{G}_r$, $\mathcal{G}^+$ and $\mathcal{G}^-$ by

$$\begin{aligned} \mathcal{G}_r &= \{g \in \mathcal{G} : \mathcal{R}(g) = g\} \\ \mathcal{G}^+ &= \{g \in \mathcal{G} : g \text{ is holomorphic on } \mathbb{C}^*\} \\ \mathcal{G}^- &= \{g \in \mathcal{G} : g \text{ is holomorphic near } 0 \text{ and } \infty\}. \end{aligned}$$

Remark. Given a complex extended framing E, since $E^{-1}\mathrm{d}E$ is holomorphic in λ on $\mathbb{C}^*$, we can see E as a map into $\mathcal{G}^+$.

Lemma 24. Set $\mathcal{G}^-_* = \{g \in \mathcal{G}^- : \ g(0) = 1\}$. Then $\mathcal{G}^+ \cap \mathcal{G}^-_* = \{1\}$.

Proof. If $g \in \mathcal{G}^+ \cap \mathcal{G}^-_*$, g is holomorphic on $\mathbb{P}^1$ and so is constant, since $\mathbb{P}^1$ is compact. Moreover $g(0) = 1$ whence $g = 1$. □

The basis of our action is the Birkhoff-Grothendieck decomposition:

Theorem 40. [35] The multiplication map

$$\mu : \mathcal{G}^+ \times \mathcal{G}^-_* \to \mathcal{G}, \ \ (g_+, g_-) \mapsto g_+g_-$$

is a diffeomorphism onto an open dense subset $\mathcal{U}$ of $\mathcal{G}$.

Hence, $g \in \mathcal{U}$ if and only if $g = g_+g_-$ with $g_+ \in \mathcal{G}^+$ and $g_- \in \mathcal{G}^-_*$. By Lemma 24 this decomposition is unique (when it exists).

For $g_- \in \mathcal{G}^-$, let $\mathcal{U}_{g_-}$ be the open neighborhood of 1 in $\mathcal{G}^+$ defined by: $g_+ \in \mathcal{U}_{g_-}$ if and only if there are unique $\hat{g}_+ \in \mathcal{G}^+$ and $\hat{g}_- \in \mathcal{G}^-_*$ such that

$$g_-g_+ = \hat{g}_+\hat{g}_- \tag{6.17}$$

on $\mathbb{C}^* \cap \mathrm{dom}(g_-)$. Write $g_- \star g_+$ for $\hat{g}_+$. Thus $g_- \star g_+ = g_-g_+\hat{g}_-^{-1}$.

One can prove easily the following:

Lemma 25. a) $\mathcal{U}_1 = \mathcal{G}^+$ and $1 \star g_+ = g_+$ for all $g_+ \in \mathcal{G}^+$;

b) for all $g_- \in \mathcal{G}^-_*$, $g_- \star 1 = 1$;

c) Let $g_1, g_2 \in \mathcal{G}^-_*$, $g_+ \in \mathcal{U}_{g_1}$ and suppose $g_1 \star g_+ \in \mathcal{U}_{g_2}$ so that $g_2 \star (g_1 \star g_+)$ is defined. Then, $g_+ \in \mathcal{U}_{g_2g_1}$ and $(g_2g_1) \star g_+ = g_2 \star (g_1 \star g_+)$.

Hence we conclude:

Theorem 41. $g_- \star g_+$ defines a local action of $\mathcal{G}^-_*$ on $\mathcal{G}^+$.

Now, let $E : \mathbb{C} \to \mathcal{G}^+$ be a smooth map and $g_- \in \mathcal{G}^-$. Define the (smooth) map $g_- \star E : E^{-1}(\mathcal{U}_{g_-}) \subset \mathbb{C} \to \mathcal{G}^+$ by

$$(g_- \star E)(p) = g_- \star (E(p)) .$$

The relevance of these results to our theory is contained in the following theorem:

Theorem 42. If $E : \mathbb{C} \to \mathcal{G}^+$ is a complex extended framing then so is $g_- \star E$.

Proof. To see that $g_- \star E$ is a complex extended frame, write

$$g_- E = ab ,$$

where $a = g_- \star E$ and $b : E^{-1}(\mathcal{U}_{g_-}) \subset \mathbb{C} \to \mathcal{G}^-_*$. Then

$$a^{-1}\mathrm{d}a = \mathrm{Ad}_b(E^{-1}\mathrm{d}E - b^{-1}\mathrm{d}b) , \tag{6.18}$$

so that

$$\lambda a^{-1}\mathrm{d}a = \mathrm{Ad}_b(\lambda E^{-1}\mathrm{d}E - \lambda b^{-1}\mathrm{d}b) \tag{6.19}$$

and

$$\lambda^{-1} a^{-1}\mathrm{d}a = \mathrm{Ad}_b(\lambda^{-1} E^{-1}\mathrm{d}E - \lambda^{-1} b^{-1}\mathrm{d}b) . \tag{6.20}$$

Now, all the ingredients on the right side of (6.19) are holomorphic in λ on a neighborhood of 0 so that $\lambda a^{-1}\mathrm{d}a$ is also; similarly, all the ingredients on the right side of (6.20) are holomorphic in λ on a neighborhood of ∞ so that $\lambda^{-1}a^{-1}\mathrm{d}a$ is also. Whence, a is a complex extended framing. □

From equation (6.18) we see that the $(1,0)$-part of $(a^{-1}\mathrm{d}a)_{\mathfrak{m}}$ lies on an adjoint orbit of the $(1,0)$-part of $(E^{-1}\mathrm{d}E)_{\mathfrak{m}}$. Then:

Proposition 18. $\varphi = E_1 \cdot \varphi_0$ is conformal if and only if $\tilde{\varphi} = a_1 \cdot \varphi_0$ is conformal.

6.6 Simple factors

Given $g_- \in \mathcal{G}^-$ and $g_+ \in \mathcal{G}^+$, a basic problem is to compute $g_- \star g_+$. In general, this is a Riemann-Hilbert problem and explicit solutions are not available. However, as the philosophy underlying the work of Terng and Uhlenbeck [42] suggests, there are certain elements of $\mathcal{G}^-$, the *simple factors*, for which one can explicitly perform the factorization $g_- g_+ = \hat{g}_+ \hat{g}_-$ by algebra alone.

Let L be an 1-dimensional isotropic subspace of $(\mathbb{R}^3)^{\mathbb{C}} \cong \mathbb{C}^3$: $(L, L) = 0$. Suppose that $QL \neq L$ and consider the decomposition

$$\mathbb{C}^3 = L \oplus QL \oplus L_0 \,,$$

where $L_0 = (L \oplus QL)^{\perp}$. Denote by π_L, π_{QL} and π_{L_0} the corresponding projections. For each $\alpha \in \mathbb{C}$ set

$$p_{\alpha,L}(\lambda) = \frac{\alpha - \lambda}{\alpha + \lambda}\pi_L + \pi_{L_0} + \frac{\alpha + \lambda}{\alpha - \lambda}\pi_{QL} \,.$$

Thus, $p_{\alpha,L} : \mathbb{P}^1 \setminus \{\pm\alpha\} \to \mathrm{SO}(3, \mathbb{C})$ and $p_{\alpha,L}(0) = 1$. Moreover, $p_{\alpha,L}$ is twisted. Then $p_{\alpha,L} \in \mathcal{G}^-_*$. The key to computing the dressing action of $p_{\alpha,L}$ is the following Lemma due to F. Burstall:

Lemma 26. [8] Let L and $\hat{L}$ be two 1-dimensional isotropic subspaces of $\mathbb{C}^3$ such that $QL \neq L$ and $Q\hat{L} \neq \hat{L}$. Define the homomorphisms $\gamma, \hat{\gamma} : \mathbb{C}^* \to \mathrm{SO}(3, \mathbb{C})$ by $\gamma(\lambda) = \lambda\pi_L + \pi_{L_0} + \lambda^{-1}\pi_{QL}$ and $\hat{\gamma}(\lambda) = \lambda\pi_{\hat{L}} + \pi_{\hat{L}_0} + \lambda^{-1}\pi_{Q\hat{L}}$. Let E be the germ at 0 of a map into $\mathrm{SO}(3, \mathbb{C})$. Then $\gamma E \hat{\gamma}^{-1}$ is holomorphic and invertible at 0 if and only if $\hat{L} = E^{-1}(0)L$.

Hence we have:

Proposition 19. Let E be a germ at α of a holomorphic map into $\mathrm{SO}(3, \mathbb{C})$ such that $\tau E(\lambda) = E(-\lambda)$. Suppose further that $QE^{-1}(\alpha)L \neq E^{-1}(\alpha)L$. Then: a) $p_{\alpha,E^{-1}(\alpha)L} \in \mathcal{G}^-_*$; b) $p_{\alpha,L}Ep^{-1}_{\alpha,E^{-1}(\alpha)L}$ is holomorphic and invertible at α.

Corollary 4. Denote by $<\varphi_0^{\perp}>$ the real orthogonal complement of $<\varphi_0>$ in $\mathbb{R}^3$. Then $g_+ \in \mathcal{U}_{p_{\alpha,L}}$ if and only if $g_+^{-1}(\alpha)L$ is not contained in $<\varphi_0^{\perp}>^{\mathbb{C}}$. For $g_+ \in \mathcal{U}_{p_{\alpha,L}}$ we have

$$p_{\alpha,L} \star g_+ = p_{\alpha,L}g_+p^{-1}_{\alpha,g_+^{-1}(\alpha)L} \,. \tag{6.21}$$

Proof. The eigenspaces of Q are $<\varphi_0>$ and $<\varphi_0^{\perp}>$. Hence, since $g_+^{-1}(\alpha)L$ is isotropic, $Qg_+^{-1}(\alpha)L \neq g_+^{-1}(\alpha)L$ if and only if $g_+^{-1}(\alpha)L$ is not contained in $<\varphi_0^{\perp}>^{\mathbb{C}}$.

The first part of Proposition 19 assures us that $p_{\alpha,g_+^{-1}(\alpha)L} \in \mathcal{G}^-_*$. So we just need to prove that $p_{\alpha,L} \star g_+$ given by (6.21) is an element of $\mathcal{G}^+$. Clearly $p_{\alpha,L} \star g_+$ is twisted, since it is a product of maps with this property. The holomorphicity at α follows directly from Proposition 19 and then we get holomorphicity at $\pm\alpha$ from the twisting condition. $\square$

6.7 Dressing actions vs. Bianchi-Bäcklund transforms

We shall show now that the dressing actions of these simple factors amount to the classical Bianchi-Bäcklund transforms.

Start with an everywhere non-conformal harmonic map $\varphi : \mathbb{C} \to S^2$. Let $E : \mathbb{C} \to \mathcal{G}^+$ be an extended framing associated to φ. By applying Bobenko-Sym formula to E, we get a map $F : \mathbb{C} \to \mathbb{R}^3$ such that $\mathrm{d}F = [\varphi, *\mathrm{d}\varphi]$, that is, a CGC $K = 1$ surface without umbilics, with normal φ. Assume that $E_\lambda(z_0) = 1$ for all $\lambda \in \mathbb{C}^*$.

Choose $\alpha \in \mathbb{C} \setminus \{0, \pm 1\}$. Consider the action of a simple factor $p_{\alpha,L}$ on E:

$$\hat{E} = p_{\alpha,L} \star E = p_{\alpha,L} E p_{\alpha,\hat{L}}^{-1} ,$$

where $\hat{L} = E(\alpha)^{-1} L$. Set $h = p_{\alpha,L}$ and $\hat{h} = p_{\alpha,\hat{L}}$. The Bobenko-Sym formula applied to $\hat{E}$ gives

$$-i\Big(\frac{\partial \hat{E}}{\partial \lambda}\Big)_{\lambda=1} \hat{E}_1^{-1} = -ih_1 \Big(\frac{\partial E}{\partial \lambda} E^{-1} - E\hat{h}^{-1} \frac{\partial \hat{h}}{\partial \lambda} E^{-1} + h^{-1} \frac{\partial h}{\partial \lambda}\Big)_{\lambda=1} h_1^{-1} ;$$

and so, our new (complex) CGC $K = 1$ surface equals

$$\hat{F} = F + iE_1 \hat{h}_1^{-1} \Big(\frac{\partial \hat{h}}{\partial \lambda}\Big)_{\lambda=1} E_1^{-1} , \tag{6.22}$$

up to (complex) Euclidean motions. The corresponding normal is $\hat{\varphi} = E_1 \hat{h}_1^{-1} \cdot \varphi_0$. By Proposition 18, $\hat{\varphi}$ is also everywhere non-conformal, whence $\hat{F}$ has no umbilic points.

Theorem 43. $\hat{F} \in \mathcal{B}_\beta(F)$, with β defined by

$$\frac{1}{\sinh \beta} = \frac{a(\alpha) - a(\alpha)^{-1}}{2} \qquad \cosh \beta = -\frac{a(\alpha) + a(\alpha)^{-1}}{a(\alpha) - a(\alpha)^{-1}} ,$$

where $a(\alpha) = \frac{\alpha+1}{\alpha-1}$. Moreover, any Bianchi-Bäcklund transformation of F is of the form (6.22) for some simple factor $p_{\alpha,L}$.

Proof. 1) Let $z = x+iy$ be a conformal coordinate on $\mathbb{C}$. Then $\mathrm{II}_F(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = \mathrm{II}_{\hat{F}}(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = 0$, since $\mathrm{d}F = [\varphi, *\mathrm{d}\varphi]$ and $\mathrm{d}\hat{F} = [\hat{\varphi}, *\mathrm{d}\hat{\varphi}]$. Hence, z is a curvature line coordinate with respect to F and $\hat{F}$.

2) Let us compute the length of $\hat{F} - F$. First note that

$$\left(\hat{h}^{-1}\frac{\partial \hat{h}}{\partial \lambda}\right)_{\lambda=1} = A(\alpha)\{\pi_{Q\hat{L}} - \pi_{\hat{L}}\}\,,$$

where $A(\alpha) = \frac{a(\alpha)-a(\alpha)^{-1}}{2} = \frac{2\alpha}{(\alpha-1)(\alpha+1)}$. Now:

$$\begin{aligned}(\hat{F} - F, \hat{F} - F) &= \frac{1}{2}\mathrm{tr}\,\{A(\alpha)E_1(\pi_{Q\hat{L}} - \pi_{\hat{L}})E_1^{-1}\}^2 \\ &= \frac{1}{2}A^2(\alpha)\mathrm{tr}\,\{\pi_{Q\hat{L}} - \pi_{\hat{L}}\}^2 = A^2(\alpha)\,,\end{aligned}$$

that is, $\hat{F} - F$ has constant length.

Moreover, since $[Q, \varphi_0] = 0$, we have

$$\begin{aligned}(\hat{F} - F, \varphi) &= -\frac{1}{2}\mathrm{tr}\,\{iA(\alpha)E_1(\pi_{Q\hat{L}} - \pi_{\hat{L}})E_1^{-1}E_1\varphi_0E_1^{-1}\} \\ &= -\frac{1}{2}\mathrm{tr}\,\{(\pi_{Q\hat{L}} - \pi_{\hat{L}})\varphi_0\} = 0\,.\end{aligned}$$

Similarly, one can prove that

$$(\hat{F} - F, \hat{\varphi}) = 0\,.$$

3) Recall that $\varphi_0 = \xi_{e_1}$, where e_1, e_2, e_3 is the canonical basis of $\mathbb{R}^3$ and $\xi_{e_1}(v) = e_1 \times v$. Since $\hat{L}$ is isotropic and $Q\hat{L} \neq \hat{L}$, $\hat{L}$ is generated by a vector $\hat{v}$ of the form $\hat{v} = \frac{1}{2}e_1 + \hat{a}e_2 + \hat{b}e_3$, with $\hat{a}^2 + \hat{b}^2 = -\frac{1}{4}$. Hence $e_1 = \hat{v} + Q\hat{v}$.

Now:

$$(\varphi, \hat{\varphi}) = -\frac{1}{2}\mathrm{tr}\,\{E_1\varphi_0E_1^{-1}E_1\hat{h}_1^{-1}\varphi_0\hat{h}_1E_1^{-1}\} = -\frac{1}{2}\mathrm{tr}\,\{\underbrace{\varphi_0\hat{h}_1^{-1}\varphi_0\hat{h}_1}_{\equiv\rho}\}\,.$$

Take $X \in \hat{L}_0$. Then

$$\begin{aligned}\rho(X) &= \varphi_0\hat{h}_1^{-1}\varphi_0\hat{h}_1(X) = \varphi_0\hat{h}_1^{-1}\big(\underbrace{\hat{v} \times X}_{\in\hat{L}} + \underbrace{Q\hat{v} \times X}_{\in Q\hat{L}}\big) \\ &= \varphi_0\big(a(\alpha)\hat{v} \times X + a(\alpha)^{-1}Q\hat{v} \times X\big) \\ &= a(\alpha)Q\hat{v} \times (\hat{v} \times X) + a(\alpha)^{-1}\hat{v} \times (Q\hat{v} \times X) \\ &= -\frac{a(\alpha) + a(\alpha)^{-1}}{2}X\,,\end{aligned}$$

where $a(\alpha) = \frac{\alpha-1}{\alpha+1}$. If $Y \in \hat{L}$ we have

$$
\begin{aligned}
\rho(Y) &= \varphi_0 \hat{h}_1^{-1} \varphi_0 \hat{h}_1(Y) = a(\alpha)^{-1} \varphi_0 \hat{h}_1^{-1} \varphi_0(Y) \\
&= a(\alpha)^{-1} \varphi_0 \hat{h}_1^{-1} (\underbrace{Q\hat{v} \times Y}_{\in \hat{L}_0}) = a(\alpha)^{-1} \varphi_0 (Q\hat{v} \times Y) \\
&= a(\alpha)^{-1} \underbrace{\hat{v} \times (Q\hat{v} \times X)}_{\in \hat{L}} + a(\alpha)^{-1} \underbrace{Q\hat{v} \times (Q\hat{v} \times Y)}_{\in Q\hat{L}} \\
&= -\frac{a(\alpha)^{-1}}{2} Y + a(\alpha)^{-1} \underbrace{Q\hat{v} \times (Q\hat{v} \times Y)}_{\in Q\hat{L}} .
\end{aligned}
$$

Similarly, if $Z \in Q\hat{L}$ we have

$$
\rho(Z) = -\frac{a(\alpha)}{2} Z + a(\alpha) \underbrace{\hat{v} \times (\hat{v} \times Z)}_{\in \hat{L}} .
$$

Then,

$$
\cos\sigma = (\varphi, \hat{\varphi}) = -\frac{1}{2} \operatorname{tr} \rho = \frac{a(\alpha) + a(\alpha)^{-1}}{2}
$$

and $(\varphi, \hat{\varphi})$ is constant.

So, we conclude that $\hat{F}$ is a Bianchi-Bäcklund transformation of F. Let θ be the corresponding solution to the sinh-Gordon equation. In order to find the spectral parameter β of this Bianchi-Bäcklund transformation, by Lemma 23 we can assume that

$$
iE_1 \big(\pi_{Q\hat{L}} - \pi_{\hat{L}}\big) E_1^{-1} = \cosh\theta e_1 + i \sinh\theta e_2 ,
$$

where (e_1, e_2, e_3) is the frame associated to F defined by (6.12). Then we have

$$
\frac{1}{\sinh\beta} = \frac{a(\alpha) - a(\alpha)^{-1}}{2} .
$$

Now, taking account (6.13), one can compute $\sin\sigma$:

$$
\sin\sigma = i\big(\varphi \times \hat{\varphi}, E_1(\pi_{Q\hat{L}} - \pi_{\hat{L}})E_1^{-1}\big) = i\frac{a^{-1}(\alpha) - a(\alpha)}{2} .
$$

Hence,

$$
-i\cosh\beta = \cot\sigma = i\frac{a(\alpha) + a(\alpha)^{-1}}{a(\alpha) - a(\alpha)^{-1}} .
$$

Evaluating $\hat{F} - F$ at z_0 gives

$$\hat{F}(z_0) - F(z_0) = iA(\alpha)(\pi_{QL} - \pi_L)\,.$$

For some fixed spectral parameter $\beta \in \mathbb{C} \setminus \{0\}$, one can easily check that there is $\alpha \in \setminus\{0, \pm 1\}$ such that

$$\frac{1}{\sinh \beta} = \frac{a(\alpha) - a(\alpha)^{-1}}{2}\,, \quad \cosh \beta = -\frac{a(\alpha) + a(\alpha)^{-1}}{a(\alpha) - a(\alpha)^{-1}}\,,$$

On the other hand, if L is generated by $v = \frac{1}{2}e_1 + ae_2 + be_3$, for some $a, b \in \mathbb{C}$ such that $a^2 + b^2 = -\frac{1}{4}$, a straightforward computation shows that

$$\pi_{QL} - \pi_L = \begin{pmatrix} 0 & 2a & 2b \\ -2a & 0 & 0 \\ -2b & 0 & 0 \end{pmatrix}\,,$$

and so any non-isotropic direction in $<\varphi_0^\perp>^{\mathbb{C}}$ is generated by a vector of the form $\pi_{QL} - \pi_L$. We therefore deduce from the uniqueness of solutions to the Bianchi-Bäcklund PDEs that each Bianchi-Bäcklund transformation of F amounts to the dressing action of some simple factor $p_{\alpha,L}$, up to an Euclidean motion. $\square$

6.7.1 Bianchi-Bäcklund Permutability theorem

Taking account the results we have obtained above, the Bianchi-Bäcklund Permutability theorem is a direct consequence of the following theorem due to F. Burstall:

Theorem 44. [8] Let $p_{\alpha_1,L_1}, p_{\alpha_2,L_2} \in \mathcal{G}_*^-$ with $\alpha_1^2 \neq \alpha_2^2$. Set $L_1' = p_{\alpha_2,L_2}(\alpha_1)L_1$ and $L_2' = p_{\alpha_1,L_1}(\alpha_2)L_2$. Assume that $QL_i' \neq L_i'$, $i = 1, 2$. Then

$$p_{\alpha_1,L_1'}p_{\alpha_2,L_2} = p_{\alpha_2,L_2'}p_{\alpha_1,L_1}\,.$$

6.7.2 Getting a real solution from an old one

If we want to obtain a new real CGC $K = 1$ surface from an old one, we have to perform two dressing actions with simple factors, as the classical theory suggests. However, note that we have no hope to find non-trivial elements in $\mathcal{G}_r^-$ of the form $p_{\beta,F}p_{\alpha,L}$. In fact, suppose that $p_{\beta,F}p_{\alpha,L} \in \mathcal{G}_r^-$. Then

$$\overline{p_{\beta,F}(\infty)p_{\alpha,L}(\infty)} = p_{\beta,F}(0)p_{\alpha,L}(0) = 1\,,$$

and this happens if and only if $F = L$ or $F = QL$. So, some technical adjustments must be made here.

For each pair (α, L), we introduce the holomorphic map $q_{\alpha,L} : \mathbb{P}^1 \setminus \{\pm\alpha\} \to \mathrm{SO}(3,\mathbb{C})$ defined by

$$q_{\alpha,L} = \frac{\lambda-\alpha}{\lambda+\alpha}\pi_L + \pi_{L_0} + \frac{\lambda+\alpha}{\lambda-\alpha}\pi_{QL}\,,$$

where, as before, $\alpha \in \mathbb{C}\setminus\{\pm 1\}$ and L is an isotropic line in $(\mathbb{R}^3)^{\mathbb{C}} \cong \mathbb{C}^3$ such that $QL \neq L$. Thus, $p_{\alpha,L} = q_{\alpha,L}(0)q_{\alpha,L}$ and, given an extended framing $E : \mathbb{C} \to \mathcal{G}^+$, we have

$$q_{\alpha,L} \star E = q_{\alpha,L} E q^{-1}_{\alpha,E^{-1}(\alpha)L} q^{-1}_{\alpha,E^{-1}(\alpha)L}(0) = q_{\alpha,L} E p^{-1}_{\alpha,E^{-1}(\alpha)L}\,,$$

that is, $q_{\alpha,L} \star E$ gives rise via Bobenko-Sym formula to the same CGC $K = 1$ surface as $p_{\alpha,L} \star E$, up to an Euclidean motion. Moreover, one can easily check that

$$\mathcal{R}(p_{\alpha,L})(\lambda) = \big(p^{-1}_{\alpha,L}(\frac{1}{\overline{\lambda}})\big)^* = q_{\frac{1}{\overline{\alpha}},\overline{L}}(\lambda)\,. \tag{6.23}$$

The following theorem, similar to Theorem 44, will also be useful:

Theorem 45. Let $p_{\alpha_1,L_1}, q_{\alpha_2,L_2} \in \mathcal{G}^-$, with $\alpha_1^2 \neq \alpha_2^2$. Set $L_1' = q_{\alpha_2,L_2}(\alpha_1)L_1$ and $L_2' = p_{\alpha_1,L_1}(\alpha_2)L_2$. Assume that $QL_i' \neq L_i'$, $i = 1, 2$. Then

$$p_{\alpha_1,L_1'} q_{\alpha_2,L_2} = P q_{\alpha_2,L_2'} p_{\alpha_1,L_1}\,,$$

with

$$P = q_{\alpha_2,L_2}(0) q^{-1}_{\alpha_2,L_2'}(0) \in K^{\mathbb{C}}\,,$$

where $K^{\mathbb{C}} \subset \mathrm{SO}(3,\mathbb{C})$ is the subgroup fixed by the involution τ.

Proof. Since $\alpha_1 \neq \pm\alpha_2$, we have that $q^{-1}_{\alpha_2,L_2}$ is holomorphic near α_1 and so we may apply Proposition 19 with $E = q^{-1}_{\alpha_2,L_2}$ to conclude that $p_{\alpha_1,L_1} q^{-1}_{\alpha_2,L_2} p^{-1}_{\alpha_1,L_1'}$ is holomorphic and invertible at $\pm\alpha_1$. Similarly, $q_{\alpha_2,L_2} p^{-1}_{\alpha_1,L_1} q^{-1}_{\alpha_2,L_2'}$ is holomorphic and invertible at $\pm\alpha_2$. Now contemplate

$$\underbrace{p_{\alpha_1,L_1'}(q_{\alpha_2,L_2} p^{-1}_{\alpha_1,L_1} q^{-1}_{\alpha_2,L_2'})}_{\text{holomorphic at } \pm\alpha_2} = \underbrace{(p_{\alpha_1,L_1} q^{-1}_{\alpha_2,L_2} p^{-1}_{\alpha_1,L_1'})^{-1} q^{-1}_{\alpha_2,L_2}}_{\text{holomorphic at } \pm\alpha_1}\,.$$

Thus, $p_{\alpha_1,L_1'}(q_{\alpha_2,L_2} p^{-1}_{\alpha_1,L_1} q^{-1}_{\alpha_2,L_2'})$ is holomorphic on $\mathbb{P}^1$ and so constant. Evaluating at $\lambda = 0$ gives

$$p_{\alpha_1,L_1'}(q_{\alpha_2,L_2} p^{-1}_{\alpha_1,L_1} q^{-1}_{\alpha_2,L_2'}) = P\,,$$

with $P = q_{\alpha_2,L_2}(0)q^{-1}_{\alpha_2,L'_2}(0)$, and so $p_{\alpha_1,L'_1}q_{\alpha_2,L_2} = Pq_{\alpha_2,L'_2}p_{\alpha_1,L_1}$. Finally, $P \in K^{\mathbb{C}}$ since all the simple factors are twisted. □

Lemma 27. If $P \in K^{\mathbb{C}}$ satisfies $P\overline{P} = 1$, then $P = \overline{k}^{-1}k$ for some $k \in K^{\mathbb{C}}$.

Proof. The complex group $\mathbb{C}^*$ double covers $K^{\mathbb{C}}$ via

$$z \in \mathbb{C}^* \mapsto \rho(z) = \mathrm{Ad}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \in \mathrm{Ad}_{\mathrm{SL}(2,\mathbb{C})} \cong \mathrm{SO}(3,\mathbb{C})\,.$$

Note that $\rho(\overline{z}) = \overline{\rho(z)}^{-1}$. Given $P \in K^{\mathbb{C}}$ such that $P\overline{P} = 1$, one can find $z_0 \in \mathbb{C}^*$ such that $\overline{z_0}^{-1}z_0 = 1$ and $\rho(z_0) = P$; in particular, we can fix $z_0 \in \mathbb{R}^+$. This means that there exists $u_0 \in \mathbb{C}^*$ such that $z_0 = \overline{u_0}u_0$. Hence

$$P = \rho(z_0) = \rho(\overline{u_0})\rho(u_0) = \overline{\rho(u_0)}^{-1}\rho(u_0)\,.$$

Set $k = \rho(u_0)$. Then $P = \overline{k}^{-1}k$. □

Corollary 5. Set $L_2 = \overline{L_1}$, $\alpha_2 = \frac{1}{\overline{\alpha_1}}$, $L'_2 = p_{\alpha_1,L_1}(\alpha_2)L_2$ and $L'_1 = q_{\alpha_2,L_2}(\alpha_1)L_1$. There exists $k \in K^{\mathbb{C}}$ such that $kq_{\alpha_2,L'_2}p_{\alpha_1,L_1} \in \mathcal{G}_r^-$.

Proof. First note that

$$\overline{L'_1} = \overline{q_{\alpha_2,L_2}(\alpha_1)L_1} = p_{\alpha_1,L_1}(\alpha_2)L_2 = L'_2\,. \tag{6.24}$$

Applying Theorem 45 together with equations (6.23) and (6.24), we get:

$$\mathcal{R}(q_{\alpha_2,L'_2}p_{\alpha_1,L_1}) = \mathcal{R}(P^{-1}p_{\alpha_1,L'_1}q_{\alpha_2,L_2}) = \mathcal{R}(P^{-1})q_{\alpha_2,L'_2}p_{\alpha_1,L_1}\,.$$

Now, set $u = q_{\alpha_2,L'_2}p_{\alpha_1,L_1}$. Then $P = \mathcal{R}(u)u^{-1} = \mathcal{R}(P^{-1})$. Hence $P\overline{P} = 1$. This means that we can find $k \in K^{\mathbb{C}}$ such that $P = \overline{k}^{-1}k$. Then

$$\mathcal{R}(ku) = \mathcal{R}(k)\mathcal{R}(u) = \bar{k}\mathcal{R}(P^{-1})u = \bar{k}\overline{P}^{-1}u = \bar{k}(k^{-1}\bar{k})^{-1}u = ku\,,$$

and we are done. □

Let $E : \mathbb{C} \to \mathcal{G}_r$ be an extended framing associated to an everywhere non-conformal harmonic map $\varphi : \mathbb{C} \to S^2$. Applying the Bobenko-Sym formula to E we get a real CGC $K = 1$ surface F. With the notations of Corollary 5,

$$\tilde{E} = (kq_{\alpha_2,L'_2}p_{\alpha_1,L_1}) \star E$$

is a new complex extended framing, which is real up to right multiplication by an element of $K^{\mathbb{C}}$, as we shall see explicitly in section 6.8.5. Therefore, the Bobenko-Sym formula applied to $\tilde{E}$ gives a new real CGC $K = 1$ surface, F^*. Up to an Euclidean motion, this surface is obtained out of F by applying two successive Bianchi-Bäcklund transforms to F: if $\beta_1 \in \mathbb{C} \setminus \{0\}$ is such that

$$\frac{1}{\sinh \beta_1} = \frac{a(\alpha_1) - a(\alpha_1)^{-1}}{2} \qquad \cosh \beta_1 = -\frac{a(\alpha_1) + a(\alpha_1)^{-1}}{a(\alpha_1) - a(\alpha_1)^{-1}}\,,$$

then, taking account Lemma 23 and Theorem 43, one can check that F^* belongs to

$$\mathcal{B}_{i\pi - \overline{\beta}_1}\left(\mathcal{B}_{\beta_1}(F)\right),$$

up to an Euclidean motion, which agrees with Theorem 38.

6.8 Bianchi-Bäcklund vs. Darboux transforms

We can define Bianchi-Bäcklund transformations for CMC surfaces by considering first their parallel CGC surface, applying Bianchi-Bäcklund transformations and then considering the parallel CMC surfaces to the transformed CGC surfaces. We conclude this chapter by relating Darboux and Bianchi-Bäcklund transformations for CMC surfaces.

Following F. Burstall [8], we shall describe isothermic surfaces and Darboux transforms, which are conformally invariant objects, in terms of Clifford algebras. In general, the use of Clifford algebras is quite efficient in the context of conformal geometry: it makes the action of the Möbius group on $\mathbb{R}^n$ particularly easy to understand (see [8]).

6.8.1 Clifford algebras

Let $C\ell_n$ be the Clifford algebra associated to the Euclidean space $(\mathbb{R}^n, (\cdot,\cdot))$. Thus $C\ell_n$ is an associative algebra with unit 1 of dimension 2^n which contains $\mathbb{R}^n$ subject only to the relations

$$vw + wv = -2(v, w)1\,.$$

Let M be a manifold and Ω the exterior algebra of differential forms on M. Consider the space $\Omega \otimes C\ell_n$ of $C\ell_n$-valued forms on M. Since $C\ell_n$ is an associative algebra, we may extend exterior multiplication and exterior derivative to $\Omega \otimes C\ell_n$ by using the product in $C\ell_n$ to multiply coefficients.

Lemma 28. [8] Let V be a real vector space with $\dim V \geq 2$ and $\alpha, \beta : V \to \mathbb{R}^n$ non-zero linear maps with α injective. Consider $\alpha \wedge \beta : \bigwedge^2 V \to C\ell_n$. Then $\alpha \wedge \beta = 0$ if and only if the following conditions are satisfied:

1. $\dim V = 2$;
2. There is $\lambda \in \mathbb{R}^+$ such that $(\beta, \beta) = \lambda(\alpha, \alpha)$;
3. $\operatorname{Im} \alpha = \operatorname{Im} \beta$;
4. $\det(\alpha^{-1} \circ \beta) < 0$.

Thus α and β have the same image, induce conformally equivalent inner products on V but opposite orientations.

6.8.2 Isothermic surfaces

Let $f : \Sigma \to \mathbb{R}^3$ be an immersion, with Σ a Riemann surface. Classically, a surface in $\mathbb{R}^3$ is *isothermic* if, away from umbilic points, it admits conformal curvature line coordinates. For example, CMC immersions are isothermic.

There is a second characterization of isothermic surfaces due to Christoffel [16]: an immersion $f : \Sigma \to \mathbb{R}^3$ is isothermic if and only if, away from umbilic points, there is a second immersion $f^c : \Sigma \to \mathbb{R}^3$, a *dual surface* of f, such that:

1. f and f^c have parallel tangent planes: $\mathrm{d}f(T_x\Sigma) = \mathrm{d}f^c(T_x\Sigma)$, for all $x \in \Sigma$.
2. f and f^c induce conformally equivalent metrics on Σ: $(\mathrm{d}f, \mathrm{d}f) = \lambda(\mathrm{d}f^c, \mathrm{d}f^c)$ for some $\lambda : \Sigma \to \mathbb{R}^+$.
3. $\mathrm{d}f^{-1} \circ \mathrm{d}f^c : T\Sigma \to T\Sigma$ is orientation-reversing: $\det(\mathrm{d}f^{-1} \circ \mathrm{d}f^c) < 0$.

In view of Lemma 28, these conditions have a compact formulation: viewing $\mathrm{d}f$ and $\mathrm{d}f^c$ as $C\ell_n$-valued 1-forms, they amount to

$$\mathrm{d}f \wedge \mathrm{d}f^c = 0\,.$$

Example 4. If $f : \Sigma \to \mathbb{R}^3$ is a CMC $H \neq 0$ immersion without umbilics, with Gauss map $\varphi : \Sigma \to S^2$, then $f^c = f + \frac{1}{H}\varphi$ is a dual surface (non-degenerate), which also has constant mean curvature H.

6.8.3 Darboux transforms

Darboux [17] discovered a transformation of isothermic surfaces: the surface f and its *Darboux transform* are characterized by the conditions that they have the same conformal structures and curvature lines and are the enveloping surfaces of a 2-sphere congruence.

Hertrich-Jeromin-Pedit [27] gave an alternative approach to the Darboux transforms: let $f : \sigma \to \mathbb{R}^3$ be an isothermic surface with dual $f^c : \Sigma \to \mathbb{R}^3$.

Suppose that, for some $r \in \mathbb{R}^\times$, the smooth map $g : \Sigma \to \mathbb{R}^3$ solves

$$\mathrm{d}g = rg\mathrm{d}f^c g - \mathrm{d}f\,. \tag{6.25}$$

Then $\hat{f} = f + g$ is a Darboux transform of f. The integrability condition for (6.25) is easily checked to be the isothermic condition $\mathrm{d}f \wedge \mathrm{d}f^c = 0$ so that, for any initial condition, we may locally solve (6.25) for g.

Notation. Fix a base point $o \in \Sigma$ and let $f : \sigma \to \mathbb{R}^3$ be an isothermic surface with dual $f^c : \Sigma \to \mathbb{R}^3$. We denote by $\mathcal{D}_r^v f$ the Darboux transform $\hat{f} = f + g$, where g solves (6.25) with $\hat{f}(0) = v$. If we do not wish to emphasize the initial condition, we shall simply write $\mathcal{D}_r f$

In the Clifford algebra formalism, Darboux transforms are characterized by:

Theorem 46. [8] Let $f, \hat{f} : \Sigma \to \mathbb{R}^3$ be isothermic. Set $g = \hat{f} - f$. Then $\hat{f}$ is a Darboux transform of f if and only if

$$\mathrm{d}\hat{f} \wedge g\mathrm{d}f g^{-1} = 0\,.$$

Otherwise said:

Theorem 47. Let $f, \hat{f} : \Sigma \to \mathbb{R}^3$ be isothermic immersions inducing the same conformal structure. Then $\hat{f}$ is a Darboux transform of f if and only if

$$\hat{\varphi} = -g\varphi g^{-1}\,,$$

where φ and $\hat{\varphi}$ are the Gauss maps of f and $\hat{f}$, respectively.

Proof. Suppose that $\hat{\varphi} = -g\varphi g^{-1}$. Then we have:

1. Since f and $\hat{f}$ are immersions, $\dim \operatorname{Im} \mathrm{d}\hat{f} = \dim \operatorname{Im} g\mathrm{d}f g^{-1}$.

2. By hypothesis, f and $\hat{f}$ induce the same conformal structures on Σ.

3. We have

$$(g\mathrm{d}f g^{-1}, \hat{\varphi}) = -(g\mathrm{d}f g^{-1}, g\varphi g^{-1}) = 0\,,$$

hence, $\operatorname{Im} \mathrm{d}\hat{f} = \operatorname{Im} g\mathrm{d}f g^{-1}$.

4. The frames of $\mathbb{R}^3$ given by $(\hat{f}_x, \hat{f}_y, \hat{\varphi})$ and (f_x, f_y, φ) have the same orientation. This means that $(\hat{f}_x, \hat{f}_y, \hat{\varphi})$ and $(-gf_x g^{-1}, -gf_y g^{-1}, \hat{\varphi} = -g\varphi g^{-1})$ have opposite orientations, since the linear map $C_g : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$C_g(X) = -gXg^{-1}$$

is reflection in the plane orthogonal to g, and so $\det C_g = -1$. Then we conclude that $d\hat{f}$ and $g df g^{-1}$ induce opposite orientations on Σ.

Taking account Lemma 28 and Theorem 46 we see now that $\hat{f}$ is a Darboux transform of f. Conversely, one can prove readily that $\hat{\varphi} = -g\varphi g^{-1}$ if $\hat{f}$ is a Darboux transform of f.

□

6.8.4 Darboux transforms of CMC surfaces

Let $f : \Sigma \to \mathbb{R}^3$ be a CMC H immersion without umbilics, with dual surface $f^c : \Sigma \to \mathbb{R}^3$ given by $f^c = Hf + \varphi$, where φ is the Gauss map of f. Fix $o \in \Sigma$. For a suitable initial condition, the Ricatti type equation (6.25) will produce a Darboux transform $\hat{f} = f + g$ of constant mean curvature H out of f. This is another classical result due to Bianchi:

Theorem 48. [4],[8],[27] If for some $r \in \mathbb{R}^\times$, $g(0) = v - f(0)$ satisfies

$$2r(g(0), \varphi(0)) - rH(g(0), g(0)) = 1\,, \tag{6.26}$$

then the Darboux transform $\mathcal{D}_r^v f$ is also a CMC H immersion.

6.8.5 Bianchi-Bäcklund vs. Darboux transforms

Start with an extended framing $E : \mathbb{C} \to \mathcal{G}_r$ associated to an everywhere non-conformal harmonic map $\varphi : \mathbb{C} \to S^2$. By applying Bobenko-Sym formula to E, we get a CGC $K = 1$ surface $F : \mathbb{C} \to \mathbb{R}^3$ such that $dF = [\varphi, *d\varphi]$.

Fix $\alpha_1 \in \mathbb{C} \setminus \{0, \pm 1\}$ and let L_1 be an isotropic line in $(\mathbb{R}^3)^{\mathbb{C}} \cong \mathbb{C}^3$ such that $QL_1 \neq L_1$. Set $L_2 = \overline{L_1}$, $\alpha_2 = \frac{1}{\overline{\alpha_1}}$, $L_2' = p_{\alpha_1, L_1}(\alpha_2)L_2$ and $L_1' = q_{\alpha_2, L_2}(\alpha_1)L_1$. By Corollary 5, we know that there exists $k \in K^{\mathbb{C}}$ such that $kq_{\alpha_2, L_2'}p_{\alpha_1, L_1} \in \mathcal{G}_r^-$. Then

$$\tilde{E} = (kq_{\alpha_2, L_2'}p_{\alpha_1, L_1}) \star E = kq_{\alpha_2, L_2'}p_{\alpha_1, L_1}Ep^{-1}_{\alpha_1, \tilde{L}_1}q^{-1}_{\alpha_2, \tilde{L}_2}q^{-1}_{\alpha_2, \tilde{L}_2}(0)\,,$$

where $\tilde{L}_1 = E^{-1}(\alpha_1)L_1$ and $\tilde{L}_2 = p_{\alpha_1, \tilde{L}_1}(\alpha_2)E^{-1}(\alpha_2)L_2$, is a new complex extended framing. Note that $E^{-1}(\alpha_2)L_2 = \overline{E^{-1}(\alpha_1)L_1}$, since E has values in $\mathcal{G}_r$. This means that $\tilde{L}_2 = p_{\alpha_1, \tilde{L}_1}(\alpha_2)\overline{\tilde{L}_1}$. Hence, by Corollary 5 we conclude that $\tilde{E}$ is real up to right multiplication by an element of $K^{\mathbb{C}}$,

Set $p = p_{\alpha_1, \tilde{L}_1}$ and $q = q_{\alpha_2, \tilde{L}_2}$. The Bobenko-Sym formula applied to $\tilde{E}$ gives a new real CGC $K = 1$ surface, which is given, up to an Euclidean motion, by

$$\tilde{F} - F = iA(\alpha_2)E_1 p_1^{-1}\{\pi_{Q\tilde{L}_2} - \pi_{\tilde{L}_2}\}p_1 E_1^{-1} + iA(\alpha_1)E_1\{\pi_{Q\tilde{L}_1} - \pi_{\tilde{L}_1}\}E_1^{-1} \equiv X\,.$$

Moreover,

$$\tilde{\varphi} = E_1 p_1^{-1} q_1^{-1} \cdot n_0 : \mathbb{C} \to S^2 \,,$$

is a new everywhere non-conformal harmonic map such that $\mathrm{d}\tilde{F} = [\tilde{\varphi}, *\mathrm{d}\tilde{\varphi}]$.

Consider the CMC $H = \frac{1}{2}$ immersions $f = F - \varphi$ and $\tilde{f} = \tilde{F} - \tilde{\varphi}$. It is an unpleasant but straightforward calculation to see that

$$\tilde{\varphi} = -g\varphi g^{-1} \,, \tag{6.27}$$

where

$$g = \tilde{f} - f = X + \varphi - \tilde{\varphi} \,,$$

whenever α_1 is real. Then, from Theorem 47 we conclude that:

Theorem 49. [27] Given a CMC surface of constant mean curvature $H = \frac{1}{2}$, any Bianchi-Bäcklund transformation $\tilde{f}$ of f associated to a real parameter β is a Darboux transform: $\tilde{f} = \mathcal{D}_r(f)$ with $r = -\frac{\sinh^2 \beta}{4}$.

Remark. Starting with a CMC $H = \frac{1}{2}$ immersion $f : \mathbb{C} \to \mathbb{R}^3$ with conformal coordinates (x, y) and fundamental forms (6.9), a Bianchi-Bäcklund transformation of f associated to a real parameter β is given by $\tilde{f} = f + g$, with

$$g = \frac{2}{\sinh \beta \cosh(\beta + \theta_1)} \{\cosh \beta e^{-\omega} (\cos \theta_2 f_x - \sin \theta_2 f_y) - \sinh \theta_2 \varphi\} \,,$$

where $\theta = \theta_1 + i\theta_2$ solves the Bianchi-Bäcklund PDEs (6.15) with respect to β and ω, and φ is the Gauss map of f. As Hertrich-Jeromin and Pedit have pointed out in [27], one can deduce Theorem 49 by showing that g solves the Ricatti equation (6.25). However, we believe that the computations we need are easier to carry out.

Remark. In order to prove (6.27), it is easier to make use of the identification

$$\mathrm{SO}(3, \mathbb{C}) \cong \mathrm{Ad}_{\mathrm{SL}(2,\mathbb{C})} \,,$$

since in this case the same computations can be performed with 2×2-matrices. The parameter r is then computed by making use of (6.26).

The situation can be described as follows: identify $\mathbb{R}^3$ with $\mathfrak{su}(2)$ via

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix} \in \mathfrak{su}(2) \,.$$

The inner product on $\mathfrak{su}(2)$ inherit from $\mathbb{R}^3$ is

$$(\xi, \eta) = -\frac{1}{2} \mathrm{tr}\, \xi\eta \,.$$

Fix

$$\varphi_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in S^2 \subset \mathfrak{su}(2)$$

and consider the automorphism τ of SU(2) given by conjugation by

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The stabilizer of φ_0, K_0, is precisely the identity component of the fixed set of τ.

Suppose now we have a complex extended framing $E : \mathbb{C}^* \to \Lambda_\tau \mathrm{SL}(2, \mathbb{C})$. In this setting, the Bobenko-Sym formula is given by

$$F = -2i \Big(\frac{\partial E}{\partial \lambda} \Big)_{\lambda=1} E_1^{-1} : \mathbb{C} \to \mathfrak{sl}(2, \mathbb{C}) ,$$

and the simple factors by

$$p_{\alpha,L}(\lambda) = \mathrm{Ad}\Big(\sqrt{\frac{\alpha - \lambda}{\alpha + \lambda}} \pi_L + \sqrt{\frac{\alpha + \lambda}{\alpha - \lambda}} \pi_{QL} \Big) ,$$

where $\alpha \in \mathbb{C} \setminus \{\pm 1\}$, $L \in \mathbb{P}^1$ is such that $QL \neq L$ and π_L, π_{QL} are the projections with respect to the decomposition $\mathbb{C}^2 = L \oplus QL$. Finally, the dressing action of $p_{\alpha,L}$ is given by

$$p_{\alpha,L} \star E = \Big(\sqrt{\frac{\alpha - \lambda}{\alpha + \lambda}} \pi_L + \sqrt{\frac{\alpha + \lambda}{\alpha - \lambda}} \pi_{QL} \Big) E \Big(\sqrt{\frac{\alpha + \lambda}{\alpha - \lambda}} \pi_{\hat{L}} + \sqrt{\frac{\alpha - \lambda}{\alpha + \lambda}} \pi_{Q\hat{L}} \Big) ,$$

where $\hat{L} = E^{-1}(\alpha) L$.

6.9 Further work

The Darboux transforms of a CMC surface with non-positive parameter r amount to the Bianchi-Bäcklund transforms with real parameter β, and these can be obtained via a certain dressing action.

1. Is there any geometrical explanation for this equivalence? Is there any relationship between Darboux transforms with positive real parameter r and Bianchi-Bäcklund transforms?[1]

[1] Added in proof: the matter has very recently been settled by J. Inoguchi and S. Kobayashi who proved that every CMC Darboux transform of a CMC surface is, in fact, a Bianchi-Bäcklund transform.

2. In [8], F. Burstall shows that the Darboux transforms of isothermic surfaces amount to the dressing action of certain simple factors. However, the underlying symmetry groups seem quite different. Hence, the problem of finding a theory of CMC surfaces that unifies the harmonic map and isothermic surface theories, formulated by F. Burstall in [8], seems now even more relevant.

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